

Modularisation of sequent calculi for normal and non-normal modalities

Björn Lellmann, Institute of Computer Languages, TU Wien, Austria

Elaine Pimentel, Departamento de Matemática, UFRN, Brazil

In this work we explore the connections between (linear) nested sequent calculi and ordinary sequent calculi for normal and non-normal modal logics. By proposing local versions to ordinary sequent rules we obtain linear nested sequent calculi for a number of logics, including to our knowledge the first nested sequent calculi for a large class of simply dependent multimodal logics, and for many standard non-normal modal logics. The resulting systems are modular and have separate left and right introduction rules for the modalities, which makes them amenable to specification as linear logic clauses. While this granulation of the sequent rules introduces more choices for proof search, we show how linear nested sequent calculi can be restricted to blocked derivations, which directly correspond to ordinary sequent derivations.

CCS Concepts: •Theory of computation → Proof theory; Modal and temporal logics; Linear logic; Automated reasoning;

Additional Key Words and Phrases: Modal logic, linear nested sequents, labelled systems, linear logic

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1. INTRODUCTION

A main concern in proof theory for modal logics is the development of philosophically and, at the same time, computationally satisfying frameworks to capture large classes of logics in a uniform and systematic way. Unfortunately the standard sequent framework satisfies these desiderata only partly. Undoubtedly, there are sequent calculi for a number of modal logics exhibiting many good properties (such as analyticity), which can be used in complexity-optimal decision procedures. However, their construction often seems ad-hoc, they are usually not modular, and they mostly lack philosophically relevant properties such as separate left and right introduction rules for the modalities. These problems are often connected to the fact that the modal rules in such calculi usually introduce more than one connective at a time. For example, in the standard presentation of the rule

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box \Gamma \Rightarrow \Box A, \Delta} \text{ k}$$

for modal logic K [Chellas 1980], the context Γ contains an arbitrary finite number of formulae, each of which is prefixed with a box in the conclusion. Hence this rule can also be seen as an infinite set of rules

$$\left\{ \frac{B_1, \dots, B_n \Rightarrow A}{\Gamma', \Box B_1, \dots, \Box B_n \Rightarrow \Box A, \Delta} \text{ k}_n \mid n \geq 0 \right\}$$

each with a fixed number of principal formulae. Both of these perspectives are somewhat dissatisfying since they require modifying the context as in the first perspective or explicitly discarding the distinction between left and right rules for the modal connective as in the second.

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One way of solving this problem is to consider extensions of the sequent framework that are expressive enough for capturing these modalities using separate left and right introduction rules. This is possible e.g. in the frameworks of *labelled sequents* [Negri and van Plato 2011] or in that of *nested sequents* or *tree-hypersequents* [Brünnler 2009; Poggiolesi 2009; Straßburger 2013]. In the labelled sequent framework the trick is accomplished by explicitly mentioning the Kripke-style relational semantics of normal modal logics in the sequents. In the nested or tree-hypersequent framework in contrast, intuitively, a single sequent is replaced with a tree of sequents, where successors of a sequent are interpreted under a modality. The modal rules of these calculi govern the transfer of (modal) formulae between the different sequents, and it can be shown that it is sufficient to transfer only one formula at a time. However, the price to pay for this added expressivity is that the obvious proof search procedure is of suboptimal complexity since it constructs potentially exponentially large nested sequents [Brünnler 2009].

In this work, we reconcile the added superior expressiveness and modularity of nested sequents with the computational behaviour of the standard sequent framework by proposing the concept of block form derivations for *linear nested sequents* [Lellmann 2015], a restricted form of nested sequents where the tree-structure is restricted to that of a line. The result is a notion of normal derivations in the linear nested setting, which directly correspond to derivations in the standard sequent setting. Moreover, the resulting calculi lend themselves to specification and implementation in linear logic following the approach in [Miller and Pimentel 2013].

Since we are interested in the connections to the standard sequent framework, we concentrate on logics which have a standard sequent calculus. Examples include normal modal logic K and simple extensions, in particular the family of simply dependent multimodal logics [Demri 2000], as well as several non-normal modal logics, i.e., standard extensions of *classical modal logic* [Chellas 1980]. Notably, we obtain the first nested sequent calculi for the logics of the *modal tesseract* (see Fig. 11).

It should be noted that some preliminary results on linear nested systems for various modal systems were presented in [Lellmann and Pimentel 2015]. In the present paper we give many more examples and refine several technical details. The new contributions with respect to [Lellmann and Pimentel 2015] are: (1) generalisation of the results on simply dependent bimodal logics to large family of logics in Sec. 3.1; (2) introduction of modular linear nested sequent calculi for the non-normal modal logics of the modal tesseract in Sec. 4; (3) definition of a notion of normal forms for linear nested sequents, via the concept of *modal block forms*; this allows for a modular way of translating modal sequent into linear nested sequent systems; (4) automatic generation of labelled systems for all the logics in the modal tesseract; and finally (5) discussion on some other possible approaches for focusing in modal systems, especially the ones proposed in [Chaudhuri et al. 2016].

The rest of the paper is organized as follows. In Section 2 we introduce the concept of linear nested sequents (LNS). In Section 3 we show that the linear nested sequent framework is a good formalism for a large class of modal systems, showing non trivial extensions of multimodal K as well as a large class of non-normal modal logics. Section 4 also presents local systems for non-normal logics, but by modifying the structural rules of the system, instead of their logical rules. In both Sections we make use of auxiliary structural operators. Since locality often entails less efficient systems, in Section 5 we propose a notion of “normal proofs” in LNS derivations, hence showing how to reduce the proof space and consequently optimize proof search. Since modal connectives presented in this work are uniquely defined by the modal rules, we can specify such rules as bipoles. We show the specification process in Section 6, by first proposing labelled sequent versions for LNS systems and then showing how to generate bipole clauses in linear logic which adequately correspond to LNS modal rules. Finally, in Section 7, we conclude by pointing out some future work.

2. LINEAR NESTED SEQUENT SYSTEMS

As an intermediate between the efficiency of the ordinary sequent framework and the expressiveness of the nested sequent framework [Brünnler 2009; Poggiolesi 2009; Straßburger 2013] we consider calculi in the *linear nested sequent* framework [Lellmann 2015]. This is essentially a reformulation of Masini’s 2-sequents [Masini 1992] in the nested sequent framework, where the tree structure

$$\begin{array}{c}
\overline{S\{\Gamma, p \Rightarrow p, \Delta\}} \text{ init} \quad \overline{S\{\Gamma, \perp \Rightarrow \Delta\}} \perp_L \quad \overline{S\{\Gamma \Rightarrow \top, \Delta\}} \top_R \quad \frac{S\{\Gamma \Rightarrow A, \Delta\}}{S\{\Gamma, \neg A \Rightarrow \Delta\}} \neg_L \quad \frac{S\{\Gamma, A \Rightarrow \Delta\}}{S\{\Gamma \Rightarrow \neg A, \Delta\}} \neg_R \\
\\
\frac{S\{\Gamma, A \Rightarrow \Delta\} \quad S\{\Gamma, B \Rightarrow \Delta\}}{S\{\Gamma, A \vee B \Rightarrow \Delta\}} \vee_L \quad \frac{S\{\Gamma, A, B \Rightarrow \Delta\}}{S\{\Gamma, A \wedge B \Rightarrow \Delta\}} \wedge_L \quad \frac{S\{\Gamma, B \Rightarrow \Delta\} \quad S\{\Gamma \Rightarrow A, \Delta\}}{S\{\Gamma, A \rightarrow B \Rightarrow \Delta\}} \rightarrow_L \\
\\
\frac{S\{\Gamma \Rightarrow A, B, \Delta\}}{S\{\Gamma \Rightarrow A \vee B, \Delta\}} \vee_R \quad \frac{S\{\Gamma \Rightarrow A, \Delta\} \quad S\{\Gamma \Rightarrow B, \Delta\}}{S\{\Gamma \Rightarrow A \wedge B, \Delta\}} \wedge_R \quad \frac{S\{\Gamma, A \Rightarrow B, \Delta\}}{S\{\Gamma \Rightarrow A \rightarrow B, \Delta\}} \rightarrow_R \\
\\
\frac{S\{\Gamma, A, A \Rightarrow \Delta\}}{S\{\Gamma, A \Rightarrow \Delta\}} \text{ ICL} \quad \frac{S\{\Gamma \Rightarrow A, A, \Delta\}}{S\{\Gamma \Rightarrow A, \Delta\}} \text{ ICR} \quad \frac{S\{\Gamma \Rightarrow \Delta\}}{S\{\Gamma, \Sigma \Rightarrow \Pi, \Delta\}} \text{ W}
\end{array}$$

Fig. 1. System LNS_G for classical propositional logic. In the init rule, p is atomic.

of nested sequents is restricted to that of a line. The benefit is that this framework exhibits the structure essential to obtain modular calculi, i.e., the nesting of sequents, while retaining a very close connection to the ordinary sequent framework and offering advantages in terms of efficiency. A similar approach was followed with the $G\text{-CK}_n$ sequents for constructive modal logic of [Mendler and Scheele 2011] which moreover also add some form of focusing to the linear structure.

In the following, we consider a *sequent* to be a pair $\Gamma \Rightarrow \Delta$ of multisets and adopt the standard conventions and notations (see e.g. [Negri and van Plato 2011]). A linear nested sequent is simply a finite list of sequents. As noted in [Lellmann 2015], this data structure matches exactly that of a *history* in a backwards proof search in an ordinary sequent calculus, a fact we will heavily use in what follows.

Definition 2.1. The set LNS of linear nested sequents is given recursively by:

- (1) if $\Gamma \Rightarrow \Delta$ is a sequent then $\Gamma \Rightarrow \Delta \in \text{LNS}$
- (2) if $\Gamma \Rightarrow \Delta$ is a sequent and $\mathcal{G} \in \text{LNS}$ then $\Gamma \Rightarrow \Delta // \mathcal{G} \in \text{LNS}$.

We will write $S\{\Gamma \Rightarrow \Delta\}$ for denoting a *context* $\mathcal{G} // \Gamma \Rightarrow \Delta // \mathcal{H}$ where each of \mathcal{G}, \mathcal{H} is a linear nested sequent or empty (omitting the $//$ symbol in the latter case). We will represent by $\vdash_C \mathcal{G}$ the fact that \mathcal{G} has a proof in the linear nested system C . We call each sequent in a linear nested sequent a *component* and slightly abuse notation and abbreviate “linear nested sequent” to LNS . The standard interpretation for linear nested sequents for modal logic K is given by:

$$\begin{aligned}
\iota_{\Box}(\Gamma \Rightarrow \Delta) &:= \bigwedge \Gamma \rightarrow \bigvee \Delta \\
\iota_{\Box}(\Gamma \Rightarrow \Delta // \mathcal{G}) &:= \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Box \iota_{\Box}(\mathcal{G})
\end{aligned}$$

I.e., the nesting is interpreted as a structural connective for the modal box.

In this work we consider only modal logics based on classical propositional logic, and we take the system LNS_G (Fig. 1) as our base calculus. It is worth noticing that the approach presented here could be easily adapted for having LKF [Liang and Miller 2009] as the base logic, since such a decision would not alter the proof theory developed for the modal connectives. Choosing LNS_G has the advantage that the initial sequents are atomic, weakening and cut are admissible and all propositional rules are invertible.

Fig. 2 presents the modal rules for the linear nested sequent calculus LNS_K for K , essentially a linear version of the standard nested sequent calculus from [Brünnler 2009; Poggiolesi 2009].

Conceptually, the main point is that the sequent rule k is split into the two rules \Box_L and \Box_R , which permit to simulate the sequent rule treating one formula at a time. While this is one of the main features of nested sequent calculi and deep inference in general [Guglielmi and Straßburger 2001],

$$\frac{S\{\Gamma \Rightarrow \Delta // \Sigma, A \Rightarrow \Pi\}}{S\{\Gamma, \Box A \Rightarrow \Delta // \Sigma \Rightarrow \Pi\}} \Box_L \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta // \Rightarrow A}{\mathcal{G} // \Gamma \Rightarrow \Delta, \Box A} \Box_R$$

Fig. 2. The modal rules of the linear nested sequent calculus LNS_K for K .

being able to separate the left/right behaviour of the modal connectives is the key to modularity for nested and linear nested sequent calculi [Straßburger 2013; Lellmann 2015].

Completeness of LNS_K w.r.t. modal logic K is shown by simulating a sequent derivation bottom-up in the last two components of the linear nested sequents, marking applications of transitional rules by the nesting $//$ and simulating the k -rule by a block of \Box_L and \Box_R rules [Lellmann 2015]. Hence, an application of k on a branch with history captured by the LNS \mathcal{G} is simulated by:

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box \Gamma \Rightarrow \Box A, \Delta} k \quad \rightsquigarrow \quad \frac{\mathcal{G} // \Gamma' \Rightarrow \Delta // \Gamma \Rightarrow A}{\mathcal{G} // \Gamma', \Box \Gamma \Rightarrow \Delta // \Rightarrow A} \Box_L \quad \frac{\mathcal{G} // \Gamma', \Box \Gamma \Rightarrow \Delta // \Rightarrow A}{\mathcal{G} // \Gamma', \Box \Gamma \Rightarrow \Box A, \Delta} \Box_R$$

where the double line indicates multiple rule applications. Observe that this method relies on the view of linear nested sequents as histories in proof search. It also simulates the propositional sequent rules in the *rightmost* component of the linear nested sequents. However, while the principal formulae of the sequent rule can now be handled separately, the modal rules in the LNS system do not need to occur in a block corresponding to one application of the sequent rule anymore. In fact, one way of deriving the instance $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ of the normality axiom for modal logic K is as follows.

$$\frac{\frac{\frac{\frac{\Box p \Rightarrow //q \Rightarrow q}{\Box p \Rightarrow //p \rightarrow q \Rightarrow q} \text{init}}{\Box(p \rightarrow q), \Box p \Rightarrow \Box q} \Box_R, \Box_L}{\Rightarrow \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)} \rightarrow_R$$

Note that the propositional rule \rightarrow_L is applied between two modal rules. Hence there are many derivations in LNS_K which are not the result of simulating a derivation of the sequent calculus for K . Thus, while the linear nested sequent calculus LNS_K has conceptual advantages over the standard sequent calculus for K , its behaviour in terms of proof search is worse: there are many more possible derivations with the same conclusion, when compared to the sequent calculus. In Section 6, we will consider how to restrict proof search to a smaller class of derivations, while retaining the conceptual advantages of the framework.

3. LINEAR NESTED SEQUENT SYSTEMS AND MODALITY

In [Lellmann 2015] the method of granularizing sequent rules into linear nested sequent rules was applied to some basic modal logics and to the multi-succedent calculus for intuitionistic logic. In the following, we will considerably extend these results and show that the linear nested sequent framework is a *good* formalism for a large class of modal systems. We first concentrate on non trivial extensions of multimodal K (the so called simply dependent multimodal logics) and then we show how to modularly extend the LNS approach also for handling non-normal modal logics.

3.1. Simply dependent multimodal logics

As a first example we consider multimodal logics with a simple interaction between the modalities, called *simply dependent multimodal logics* [Demri 2000]. The language for these logics contains indexed modalities \Box_i for indices i from an index set $N \subseteq \mathbb{N}$ of natural numbers. The axioms are given by extensions of the axioms of modal logic K for every modality \Box_i together with *interaction axioms* of the form $\Box_i A \rightarrow \Box_j A$. A simple example of such a logic is the simply dependent bimodal

$$\text{K } \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \quad \frac{A}{\Box A} \text{ nec} \quad \text{D } \Box \neg \Box \perp \quad \text{T } \Box A \rightarrow A \quad 4 \Box A \rightarrow \Box \Box A$$

Fig. 3. Standard modal axioms and rule. Modal logic K contains the propositional tautologies, modus ponens, K and nec.

logic $\text{KT} \oplus_{\subseteq} \text{S4}$ from [Demri 2000], whose language contains the two modalities \Box_1 and \Box_2 , the KT axioms for the modality \Box_1 as well as the S4 axioms for the modality \Box_2 and the single interaction axiom $\Box_2 A \rightarrow \Box_1 A$. Standard modal logics such as K or extensions also can be seen as the trivial case of simply dependent multimodal logics where the index set N is a singleton. Other examples include multimodal logics with a *justified knowledge* or “any fool knows” modality from [Artemov 2006; McCarthy et al. 1978].

A general framework to describe simply dependent multimodal logics was given in [Achilleos 2016, Sec. 4]. There such a logic is given essentially by a triple (N, \leq, F) , where N is a finite set of natural numbers, (N, \leq) is a partial order (i.e., transitive, reflexive and antisymmetric), and F is a mapping from N to a set \mathcal{L} of logics.

In the present work, we will take \mathcal{L} to be the set of extensions of modal logic K with axioms from the set $\{\text{D}, \text{T}, 4\}$ (see Fig. 3). The logic described by (N, \leq, F) then has modalities \Box_i for every $i \in N$, with axioms for the modality i given by the logic $F(i)$ and interaction axioms $\Box_j A \rightarrow \Box_i A$ for every $i, j \in N$ with $i \leq j$.

Example 3.1. The simply dependent bimodal logic $\text{KT} \oplus_{\subseteq} \text{S4}$ is given by the description $((1, 2), \{(1, 1), (1, 2), (2, 2)\}, F)$ where $F(1) = \text{KT}$ and $F(2) = \text{S4}$.

The following definition extends the concept of frames to simply dependent multimodal logic. The notions of valuations, model and truth in a world of the model are defined as usual, see, e.g., [Chellas 1980; Blackburn et al. 2001].

Definition 3.2. Let (N, \leq, F) be a description for a simply dependent multimodal logic. A (N, \leq, F) -frame is a tuple $(W, (R_i)_{i \in N})$ consisting of a set W of worlds and an accessibility relation R_i for every index $i \in N$, such that for all $i, j \in N$:

- If the logic $F(i)$ contains KD, then R_i is serial.
- If the logic $F(i)$ contains KT, then R_i is reflexive.
- If the logic $F(i)$ contains K4, then R_i is transitive.
- If $i \leq j$, then $R_i \subseteq R_j$.

Since here we only consider simply dependent multimodal logics where the different component logics are extensions of K with axioms from $\{\text{D}, \text{T}, 4\}$, and since the interaction axioms are of a particularly simple shape, standard results e.g. from Sahlqvist theory [Blackburn et al. 2001] immediately yield completeness:

THEOREM 3.3. *The modal logic given by the description (N, \leq, F) is the logic of the class of (N, \leq, F) -frames. \square*

By standard modal reasoning we immediately obtain the following lemma stating upwards propagation of the modal axioms D and T.

LEMMA 3.4. *Let (N, \leq, F) be a description of a simply dependent multimodal logic \mathcal{L} . Then for every $i \in N$:*

- *If $\text{KD} \subseteq F(i)$, then for every $j \in N$ with $i \leq j$, $\neg \Box_j \perp$ is also a theorem of \mathcal{L} .*
- *If $\text{KT} \subseteq F(i)$, then for every $j \in N$ with $i \leq j$, $\Box_j A \rightarrow A$ is also a theorem of \mathcal{L} .*

PROOF. By closing the axioms under the interaction axioms $\Box_j A \rightarrow \Box_i A$ for $i \leq j$. Alternatively, this can be seen from the semantical characterisation. \square

$$\begin{array}{c}
\frac{\{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{-4}(i)\} \Rightarrow A}{\Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j \Sigma_j : j \in \uparrow^{-4}(i)\} \Rightarrow \Box_i A, \Xi} k_i \\
\\
\frac{\{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{-4}(i)\} \Rightarrow}{\Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j \Sigma_j : j \in \uparrow^{-4}(i)\} \Rightarrow \Xi} d_i \quad \frac{\Omega, \{\Box_j \Sigma_j, \Sigma_j : j \in \uparrow(i)\} \Rightarrow \Xi}{\Omega, \{\Box_j \Sigma_j : j \in \uparrow(i)\} \Rightarrow \Xi} t_i \\
\\
\mathcal{R}_{(N, \leq, F)} := \{k_i : i \in N\} \cup \{d_i : i \in N, KD \subseteq F(i)\} \cup \{t_i : i \in N, KT \subseteq F(i)\}
\end{array}$$

Fig. 4. The modal sequent rules for the simply dependent multimodal logic given by the transitive-closed description (N, \leq, F) .

Hence we may assume, without loss of generality, that for any description (N, \leq, F) and any $i \in N$, if $KD \subseteq F(i)$ (or $KT \subseteq F(i)$), then for every $j \in N$ with $i \leq j$ we have $KD \subseteq F(j)$ (resp. $KT \subseteq F(j)$). As a more economic notation we also write $\uparrow(i)$ for the *upset* of the index i , i.e., the set $\{j \in N : i \leq j\}$. Furthermore, in light of the comments above we extend this notation to the sets $\uparrow^{Ax}(i) := \{j \in N : i \leq j, KAx \subseteq F(j)\}$ and $\uparrow^{-Ax}(i) := \{j \in N : i \leq j, KAx \not\subseteq F(j)\}$ where Ax is any of the axioms D, T, 4. Thus e.g. the set $\uparrow^{-4}(i)$ is the set of indices j with $i \leq j$ such that $K4 \not\subseteq F(j)$, i.e., the logic $F(j)$ does not derive the transitivity axiom 4.

The next step is to obtain cut-free sequent calculi for logics of this shape. In order to obtain cut-free completeness we need to stipulate an additional condition on the description, stating that the set of transitive logics is upwards closed. Formally:

Definition 3.5. A description (N, \leq, F) is *transitive-closed* if for every $i, j \in N$, $i \leq j$, if $K4 \subseteq F(i)$ then $K4 \subseteq F(j)$.

Using e.g. the method of [Lellmann and Pattinson 2013; Lellmann 2013] it is then straightforward to construct cut-free sequent calculi for simply dependent multimodal logics given by a transitive-closed description. The resulting modal rules and rule sets are given in Fig. 4.

Definition 3.6. The restriction of the propositional calculus LNS_G from Fig. 1 to sequents is denoted by G . If (N, \leq, F) is a description for a simply dependent multimodal logic, then $G_{(N, \leq, F)}$ is the sequent calculus extending the propositional calculus G with the modal rules $\mathcal{R}_{(N, \leq, F)}$ according to Fig. 4.

Example 3.7. In the case of our example logic $KT \oplus_{\subseteq} S4$ we have $KD \subseteq KT \subseteq F(i)$ for $i = 1, 2$ and $K4 \subseteq F(2)$ but $K4 \not\subseteq F(1)$. Hence the sequent calculus $G_{KT \oplus_{\subseteq} S4}$ for this logic contains the following modal rules:

$$\begin{array}{c}
\frac{\Box_2 \Gamma, \Box_2 \Sigma, \Sigma, \Theta \Rightarrow A}{\Omega, \Box_2 \Gamma, \Box_2 \Sigma, \Box_1 \Theta \Rightarrow \Box_1 A, \Xi} k_1 \quad \frac{\Box_2 \Gamma \Rightarrow A}{\Omega, \Box_2 \Gamma \Rightarrow \Box_2 A, \Xi} k_2 \\
\\
\frac{\Box_2 \Gamma_2, \Box_2 \Sigma_2, \Sigma_2, \Sigma_1 \Rightarrow}{\Omega, \Box_2 \Gamma_2, \Box_2 \Sigma_2, \Box_1 \Sigma_1 \Rightarrow \Theta} d_1 \quad \frac{\Box_2 \Gamma_2, \Box_2 \Sigma_2, \Sigma_2 \Rightarrow}{\Omega, \Box_2 \Gamma_2, \Box_2 \Sigma_2 \Rightarrow \Theta} d_2 \quad \frac{\Gamma, \Box_1 \Sigma, \Sigma \Rightarrow \Delta}{\Gamma, \Box_1 \Sigma \Rightarrow \Delta} t_1 \quad \frac{\Gamma, \Box_2 \Sigma, \Sigma \Rightarrow \Delta}{\Gamma, \Box_2 \Sigma \Rightarrow \Delta} t_2
\end{array}$$

Note that the rules d_1, d_2 are derivable using rules t_1, t_2 , and hence could be omitted from the rule set. For the sake of a uniform presentation we decided to keep them.

While soundness and completeness of the calculi $G_{(N, \leq, F)}$ follow directly from the construction, it is worth stating them explicitly.

THEOREM 3.8. Let (N, \leq, F) be a description of a simply dependent multimodal logic. Then the sequent calculus $G_{(N, \leq, F)}$ is sound and cut-free complete with respect to this logic.

PROOF. For soundness we use the fact that the logic given by the description is also characterised by frames $(W, (R_i)_{i \in N})$ where for $i \in N$ the accessibility relation R_i satisfies the properties stipulated

by the logic $F(i)$ (i.e., is serial if $KD \subseteq F(i)$, reflexive if $KT \subseteq F(i)$ and transitive if $K4 \subseteq F(i)$), and where for every $i, j \in N$ with $i \leq j$ we have $R_i \subseteq R_j$. Then it is straightforward to show that all the modal rules preserve validity by showing that if the negation of the conclusion is satisfiable in such a frame, then so is the premiss. As an example we fix a description (N, \leq, F) and consider the following application of the rule d_i for an index $i \in N$ such that $F(i)$ is serial.

$$\frac{\{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^4(i)\} \Rightarrow}{\Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j \Sigma_j : j \in \uparrow^4(i)\} \Rightarrow \Xi} d_i$$

If the negation of the conclusion of this rule is satisfiable in a (N, \leq, F) -model $\mathfrak{M} = (W, (R_i)_{i \in N}, \sigma)$, then we have a world $w \in W$ such that

$$\mathfrak{M}, w \Vdash \bigwedge \Omega \wedge \bigwedge_{j \in \uparrow^4(i)} (\bigwedge \Box_j \Gamma_j \wedge \bigwedge \Box_j \Sigma_j) \wedge \bigwedge_{j \in \uparrow^4(i)} \bigwedge \Box_j \Sigma_j \wedge \neg \bigvee \Xi. \quad (1)$$

Since $F(i)$ is serial, there is a world $v \in W$ with wR_iv , and since $i \leq j$ implies $R_i \subseteq R_j$ for all $i, j \in N$, for this v we also have wR_jv for every j with $i \leq j$. Hence using (1) and transitivity of the relations R_j for j with $K4 \subseteq F(j)$ we obtain

$$\mathfrak{M}, v \Vdash \bigwedge_{j \in \uparrow^4(i)} (\bigwedge \Box_j \Gamma_j \wedge \bigwedge \Box_j \Sigma_j \wedge \bigwedge \Sigma_j) \wedge \bigwedge_{j \in \uparrow^4(i)} \bigwedge \Sigma_j.$$

Hence the negation of the interpretation of the premiss of this rule application is satisfied in v . The reasoning for the remaining rules is similar.

For cut-free completeness, first it is a simple exercise to derive all the axioms for \mathcal{L} in the system with the cut rule. Then a standard argument shows that the system has cut elimination (see e.g. the general criteria for cut elimination in [Lellmann and Pattinson 2013; Lellmann 2013]).

To illustrate the reasoning consider, as an example, the cut below, with applications of rules based on a description such that $i \leq j, k$ and $\ell \leq i, m, n$ with $K4 \subseteq F(j), F(m)$ but with $K4$ not contained in the other logics.

$$\frac{\frac{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j, \Sigma_k, \Sigma_i \Rightarrow A}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_k \Sigma_k, \Box_i \Sigma_i \Rightarrow \Box_i A} k_i \quad \frac{\Box_m \Gamma_m, \Box_m \Sigma_m, \Sigma_m, A, \Sigma_i, \Sigma_n \Rightarrow}{\Box_m \Gamma_m, \Box_m \Sigma_m, \Box_i A, \Box_i \Sigma_i, \Box_n \Sigma_n \Rightarrow} d_\ell}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_k \Sigma_k, \Box_i \Sigma_i, \Box_m \Gamma_m, \Box_m \Sigma_m, \Box_i \Sigma_i, \Box_n \Sigma_n \Rightarrow} \text{cut}$$

As usual, the cut is replaced with a cut on the formula of lower complexity A as follows.

$$\frac{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j, \Sigma_k, \Sigma_i \Rightarrow A \quad \Box_m \Gamma_m, \Box_m \Sigma_m, \Sigma_m, A, \Sigma_i, \Sigma_n \Rightarrow}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j, \Sigma_k, \Sigma_i, \Box_m \Gamma_m, \Box_m \Sigma_m, \Sigma_m, \Sigma_i, \Sigma_n \Rightarrow} \text{cut} \quad \frac{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_k \Sigma_k, \Box_i \Sigma_i, \Box_m \Gamma_m, \Box_m \Sigma_m, \Box_i \Sigma_i, \Box_n \Sigma_n \Rightarrow}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_k \Sigma_k, \Box_i \Sigma_i, \Box_m \Gamma_m, \Box_m \Sigma_m, \Box_i \Sigma_i, \Box_n \Sigma_n \Rightarrow} d_\ell$$

Crucially, since the relation \leq is transitive, we know that $\ell \leq j, k$ as well, which renders the application of the rule d_ℓ at the bottom permissible.

As a second example, consider the cut with cut formula $\Box_i A$ below, based on a description (N, \leq, F) with $k \leq i \leq j$, such that $K4 \subseteq F(i), F(j)$ and $KD \subseteq F(k)$.

$$\frac{\frac{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j \Rightarrow A}{\Box_j \Gamma_j, \Box_j \Sigma_j \Rightarrow \Box_i A} k_i \quad \frac{\Box_i A, \Box_i \Gamma_i, \Box_i \Sigma_i, \Sigma_i, \Gamma_k \Rightarrow}{\Box_i A, \Box_i \Gamma_i, \Box_i \Sigma_i, \Box_k \Gamma_k \Rightarrow} d_k}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_i \Gamma_i, \Box_i \Sigma_i, \Box_k \Gamma_k \Rightarrow} \text{cut}$$

This cut is reduced to a cut of lower height by permuting it into the premiss of the application of d_k :

$$\frac{\frac{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j \Rightarrow A}{\Box_j \Gamma_j, \Box_j \Sigma_j \Rightarrow \Box_i A} k_i \quad \Box_i A, \Box_i \Gamma_i, \Box_i \Sigma_i, \Sigma_i, \Gamma_k \Rightarrow}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_i \Gamma_i, \Box_i \Sigma_i, \Sigma_i, \Gamma_k \Rightarrow} \text{cut} \quad \frac{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_i \Gamma_i, \Box_i \Sigma_i, \Sigma_i, \Gamma_k \Rightarrow}{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_i \Gamma_i, \Box_i \Sigma_i, \Box_k \Gamma_k \Rightarrow} d_k$$

$$\begin{array}{c}
\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta //^j \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box_i A \Rightarrow \Delta //^j \Sigma \Rightarrow \Pi\}} \Box_{iL} \quad \frac{\mathcal{G} //^k \Gamma \Rightarrow \Delta //^i \Rightarrow A}{\mathcal{G} //^k \Gamma \Rightarrow \Delta, \Box_i A} \Box_{iR} \\
\\
\frac{\mathcal{G} //^k \Gamma \Rightarrow \Delta //^j A \Rightarrow}{\mathcal{G} //^k \Gamma, \Box_i A \Rightarrow \Delta} \mathbf{d}_{ij} \quad \frac{\mathcal{S}\{\Gamma, \Box_i A, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Box_i A \Rightarrow \Delta\}} \mathbf{t}_i \quad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta //^j \Sigma, \Box_i A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box_i A \Rightarrow \Delta //^j \Sigma \Rightarrow \Pi\}} 4_{ij} \\
\\
\mathcal{R}_{(N, \leq, F)} := \{\Box_{iR} : i \in N\} \cup \{\Box_{iL} : i, j \in N, i \in \uparrow(j)\} \cup \{\mathbf{d}_{ij} : i, j \in N, i \in \uparrow^D(j)\} \\
\cup \{\mathbf{t}_i : i \in N, \mathbf{KT} \subseteq F(i)\} \cup \{4_{ij} : i, j \in N, i \in \uparrow^4(j)\}
\end{array}$$

Fig. 5. The linear nested sequent rules for the simply dependent multimodal logic given by the description (N, \leq, F) .

Note that for this transformation to work it is crucial that the logic $F(j)$ is also transitive, since otherwise we would not be able to apply the rule \mathbf{d}_k with boxed context formulae $\Box_j \Gamma_j, \Box_j \Sigma_j$. A similar situation occurs if the application of \mathbf{d}_k is replaced with an application of \mathbf{k}_k with an additional principal formula on the right.

The general cases of the above examples as well as the remaining cases then are treated similarly. \square

In order to convert the resulting sequent systems into LNS systems, we need to modify the linear nested setting to account for all the different non-invertible right rules. For this, given a description (N, \leq, F) we introduce nesting operators $//^i$ for every $i \in N$, and change the interpretation so that they are interpreted by the corresponding modality:

$$\begin{aligned}
\iota(\Gamma \Rightarrow \Delta) &:= \bigwedge \Gamma \rightarrow \bigvee \Delta \\
\iota(\Gamma \Rightarrow \Delta //^i \mathcal{H}) &:= \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Box_i \iota(\mathcal{H})
\end{aligned}$$

The modal sequent rules of $\mathbf{G}_{(N, \leq, F)}$ are then decomposed into the modal linear nested sequent rules shown in Fig. 5. The propositional rules are those of $\mathbf{LNS}_{\mathbf{G}}$ (Fig. 1). We call the resulting calculus $\mathbf{LNS}_{(N, \leq, F)}$.

Example 3.9. The linear nested sequent calculus for the logic $\mathbf{KT} \oplus_{\subseteq} \mathbf{S4}$ contains the LNS rules $\Box_{11L}, \Box_{21L}, \Box_{22L}, \Box_{1R}, \Box_{2R}, \mathbf{d}_{11}, \mathbf{d}_{21}, \mathbf{d}_{22}, \mathbf{t}_1, \mathbf{t}_2$ and 4_{21} .

THEOREM 3.10. *If (N, \leq, F) is a description of a simply dependent multimodal logic, then $\mathbf{LNS}_{(N, \leq, F)}$ is sound and complete for the logic given by (N, \leq, F) .*

PROOF. For soundness we show that whenever the negation of the interpretation of the conclusion of a rule from $\mathbf{LNS}_{(N, \leq, F)}$ is satisfiable in a (N, \leq, F) -frame, then so is the negation of the interpretation of at least one of its premisses. This makes essential use of the fact that in such frames we have $R_i \subseteq R_j$ whenever $i \leq j$. For completeness, we again simulate the sequent rules in the last components, i.e., we translate a sequent derivation in $\mathbf{G}_{(N, \leq, F)}$ bottom-up into a linear nested sequent derivation in $\mathbf{LNS}_{(N, \leq, F)}$, simulating propositional sequent rules by their linear nested sequent counterparts, and modal sequent rules by a number of applications of the corresponding linear nested sequent rules. E.g., an application of the modal sequent rule \mathbf{d}_i with history (i.e., trace to the conclusion of the sequent derivation) captured by the linear nested sequent \mathcal{G} is simulated as follows (assuming that $k \in \uparrow^{-4}(i)$):

$$\frac{\{\Box_j \Gamma_j, \Box_j \Sigma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow}{\Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j \Sigma_j : j \in \uparrow^{-4}(i)\}, \Box_k A \Rightarrow \Xi} \mathbf{d}_i$$

\vdots
 \mathcal{G}

$$\begin{array}{c}
\frac{\frac{\mathcal{G} // \Omega, \Rightarrow \Xi //^i \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow}{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \Rightarrow \Xi //^i \{\Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow} 4_{ji}}{\frac{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \Rightarrow \Xi //^i \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow}{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \Rightarrow \Xi //^i \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow} \Box_{ji_L}} \sim \\
\frac{\frac{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \Rightarrow \Xi //^i \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow}{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \Rightarrow \Xi //^i \{\Sigma_j : j \in \uparrow^{-4}(i)\}, A \Rightarrow} \Box_{ji_L}}{\frac{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j \Sigma_j : j \in \uparrow^{-4}(i)\} \Rightarrow \Xi //^i A \Rightarrow}{\mathcal{G} // \Omega, \{\Box_j \Gamma_j, \Box_j \Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j \Sigma_j : j \in \uparrow^{-4}(i)\}, \Box_k A \Rightarrow \Xi} \Box_{ki}} \text{ICL}
\end{array}$$

The remaining modal rules are simulated in a similar way. \square

Note that the proof of completeness via simulation of the sequent calculus in the last component actually shows a slightly stronger statement, i.e., completeness for a variant of the calculus where the rules are restricted so they only manipulate the last components. More precisely:

Definition 3.11. An application of a linear nested sequent rule is *end-active* if the rightmost components of the premisses are active and the only active components (in premiss and conclusion) are the two rightmost ones. The *end-active variant* of a LNS calculus is the calculus with the rules restricted to end-active applications.

Example 3.12. The application of the rule \wedge_L below left is end-active, the one below right is not, since the rightmost component is not active.

$$\frac{\mathcal{G} // \Gamma, A, B \Rightarrow \Delta}{\mathcal{G} // \Gamma, A \wedge B \Rightarrow \Delta} \wedge_L \qquad \frac{\mathcal{G} // \Gamma, A, B \Rightarrow \Delta // \Sigma \Rightarrow \Pi}{\mathcal{G} // \Gamma, A \wedge B \Rightarrow \Delta // \Sigma \Rightarrow \Pi} \wedge_L$$

Applications of the modal rules in the LNS calculi for non-normal modal logics considered in this paper (see next section) are always end-active. An application of the modal rule

$$\frac{S\{\Gamma \Rightarrow \Delta // \Sigma, A \Rightarrow \Pi\}}{S\{\Gamma, \Box A \Rightarrow \Delta // \Sigma \Rightarrow \Pi\}} \Box_L$$

is end-active only if $\Sigma \Rightarrow \Pi$ is the rightmost component.

COROLLARY 3.13. If (N, \leq, F) is a description of a simply dependent multimodal logic, then the end-active variant of $\text{LNS}_{(N, \leq, F)}$ is sound and complete for the logic given by (N, \leq, F) .

PROOF. Soundness follows immediately from soundness for the full calculus. For completeness observe that the sequent rules are simulated in the last component, i.e., by end-active applications of the linear nested sequent rules. \square

The fact that we can restrict the linear nested calculi to their end-active variants will be exploited in Section 5 for reducing the search space in proof search.

The example of simply dependent multimodal logics shows another conceptual advantage of LNS calculi over standard sequent calculi: for more involved sequent calculi such as the ones in Fig. 4 the decomposition of the sequent rules into their different components tends to make the corresponding LNS calculi (Fig. 5) a lot more readable. Of course the previous theorem also shows that the obvious adaption of this calculus to the full nested sequent setting of [Brünnler 2009; Poggiolesi 2009] is sound and cut-free complete for the corresponding logic.

3.2. Non-normal modal logics

The same ideas also yield LNS calculi for some *non-normal* modal logics, i.e., modal logics that are not extensions of modal logic K (see [Chellas 1980] for an introduction). The calculi themselves are of independent interest since, to the best of our knowledge, nested sequent calculi for the logics below have not been considered before in the literature. The most basic non-normal logic, *classical modal logic E*, is given Hilbert-style by extending the axioms and rules for classical propositional

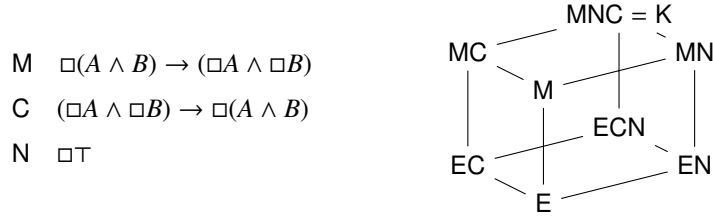
$\frac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \text{ (E)}$	$\frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \text{ (M)}$	$\frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta} \text{ (N)}$
$\frac{A_1, \dots, A_n \Rightarrow B \quad B \Rightarrow A_1 \quad \dots \quad B \Rightarrow A_n}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \text{ (En)}$	$\frac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \text{ (Mn)}$	
$G_E \quad \{(E)\}$	$G_{EN} \quad \{(E), (N)\}$	$G_{EC} \quad \{(En) : n \geq 1\}$
$G_M \quad \{(M)\}$	$G_{MN} \quad \{(M), (N)\}$	$G_{MC} \quad \{(Mn) : n \geq 1\}$
		$G_{ECN} \quad \{(En) : n \geq 1\} \cup \{(N)\}$
		$G_{MCN} \quad \{(Mn) : n \geq 0\}$

Fig. 6. Sequent rules and calculi for some non-normal modal logics

logic with only the *congruence rule* (E) for the \Box connective

$$\frac{A \rightarrow B \quad B \rightarrow A}{\Box A \rightarrow \Box B} (E)$$

which allows exchanging logically equivalent formulae under the modality. Some of the better known extensions of this logic are formulated by the addition of axioms from the list below left.



Together, these extensions form what could be termed the *classical cube* (above right). Note that the extension of E with all the axioms M, C, N is normal modal logic K. Fig. 6 shows the modal rules of the standard cut-free sequent calculi for these logics, where in addition weakening is embedded in the conclusion. For all of the logics apart from EN and ECN these calculi were given in [Lavendhomme and Lucas 2000], the remaining calculi are easy extensions. Extensions of E are written by concatenating the names of the axioms, and in presence of the monotonicity axiom M, sometimes the initial E is dropped. E.g., the logic EMC = MC is the extension of E with axioms M and C. Its sequent calculus G_{MC} is given by the standard propositional and structural rules of G (see Def. 3.6) together with the rule (E) as well as the rules (Mn) for $n \geq 1$.

In order to construct linear nested calculi for these logics, again we would like to decompose the sequent rules from Fig. 6 into their different components. However, there are two complications compared to the case of normal modal logics: we need a mechanism for capturing the fact that e.g. in the rule (M) exactly one boxed formula is introduced on the left hand side; and we need a way of handling multiple premisses of rules such as (E) and (En). We solve the first problem by introducing an auxiliary nesting operator $\|_e$ to capture a state where a sequent rule has been *partly processed*. In contrast, the intuition for the original nesting $\|$ is that the simulation of a rule is finished. We restrict the occurrence of $\|_e$ to the end of the structures.

To solve the problem of multiple premisses, we make the nesting operator $\|_e$ *binary*, which permits the storage of more information about the premisses. In particular, we can now store the two “directions” of implications given, e.g., in the premisses of rule (E). Linear nested sequents for classical non normal modal logics are then given by:

$$LNS_e ::= \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \|_e (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta) \mid \Gamma \Rightarrow \Delta \| LNS_e$$

Fig. 7 shows the modal rules for these logics. For a logic $E\mathcal{A}$ with $\mathcal{A} \subseteq \{N, M, C\}$ the calculus $LNS_{E\mathcal{A}}$ then contains the corresponding modal rules along with the propositional rules of LNS_G (Fig. 1) with

$$\begin{array}{c}
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B \Rightarrow)}{\mathcal{G} // \Gamma \Rightarrow \Box B, \Delta} \Box_R^e \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta // \Sigma, A \Rightarrow \Pi \quad \mathcal{G} // \Gamma \Rightarrow \Delta // \Omega \Rightarrow A, \Theta}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\mathbf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \Box_L^e \\
\\
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta // \Rightarrow B}{\mathcal{G} // \Gamma \Rightarrow \Box B, \Delta} \mathbf{N} \quad \frac{\mathcal{G} //_{\mathbf{e}} (\Sigma \Rightarrow \Pi; \Omega, \perp \Rightarrow \Theta)}{\mathcal{G} //_{\mathbf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \mathbf{M} \\
\\
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\mathbf{e}} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta) \quad \mathcal{G} // \Gamma \Rightarrow \Delta // \Omega \Rightarrow A, \Theta}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\mathbf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \mathbf{C} \\
\\
\text{LNS}_{\mathcal{E}, \mathcal{A}} : \quad \{\Box_R, \Box_L\} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\mathbf{N}, \mathbf{M}, \mathbf{C}\}
\end{array}$$

Fig. 7. Linear nested sequent calculi for non-normal modal logics

the restriction that they are not applied inside the nesting $//_{\mathbf{e}}$. To keep the presentation simple, we slightly abuse notation and write e.g. \mathbf{M} both for the axiom and the corresponding rule.

THEOREM 3.14 (COMPLETENESS). *The linear nested sequent calculi of Fig. 7 are complete w.r.t. the corresponding logics.*

PROOF. Again the proof is via simulation of the sequent calculi. An application of the rule (\mathbf{En}) is simulated by the derivation

$$\begin{array}{c}
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta // A_1, \dots, A_n \Rightarrow B \quad \mathcal{G} // \Gamma \Rightarrow \Delta // B \Rightarrow A_n}{\mathcal{G} // \Gamma, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (A_1, \dots, A_{n-1} \Rightarrow B; B \Rightarrow)} \Box_L^e \\
\vdots \\
\frac{\mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (A_1 \Rightarrow B; B \Rightarrow) \quad \mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta // B \Rightarrow A_1}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B \Rightarrow)} \mathbf{C} \\
\frac{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \Box_R^e
\end{array}$$

The case of $n = 1$ gives the simulation of the rule (\mathbf{E}) . The sequent rule \mathbf{N} is simulated directly by the LNS rule \mathbf{N} . In the monotone case the simulations are essentially the same, but after creating the new nesting using the \Box_R^e rule we first apply the rule \mathbf{M} to make all the premisses for the “backwards direction” trivially derivable. The sequent rule (\mathbf{Mn}) then is simulated by

$$\begin{array}{c}
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta // A_1, \dots, A_n \Rightarrow B \quad \mathcal{G} // \Gamma \Rightarrow \Delta // B, \perp \Rightarrow A_n}{\mathcal{G} // \Gamma, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (A_1, \dots, A_{n-1} \Rightarrow B; B, \perp \Rightarrow)} \Box_L^e \quad \perp_L \\
\vdots \\
\frac{\mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (A_1 \Rightarrow B; B, \perp \Rightarrow) \quad \mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta // B, \perp \Rightarrow}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B, \perp \Rightarrow)} \mathbf{C} \quad \perp_L \\
\frac{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B \Rightarrow)} \mathbf{M} \\
\frac{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}} (\Rightarrow B; B \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \Box_R^e
\end{array}$$

Again, the case of $n = 1$ gives the simulation of the rule (\mathbf{M}) . \square

As in the case of simply dependent multimodal logics, the proof of completeness via simulation of the sequent rules in the last component also shows completeness of the end-active variants of the calculi.

COROLLARY 3.15. *The end-active variants of the linear nested sequent calculi of Fig. 7 are complete w.r.t. the corresponding logics.*

For showing soundness of such calculi we need a different method, though. This is due to the fact that, unlike for normal modal logics, there is no clear formula interpretation for linear nested sequents for non-normal modal logics. However, since the propositional rules cannot be applied inside the auxiliary nesting \parallel_e , the modal rules can only occur in blocks which can be seen as a macro-rule corresponding to a modal sequent rule. In addition, we will show by a permutation-of-rules argument that it is possible to restrict the propositional rules to end-active applications (see Def. 3.11). Soundness of the full calculus then follows from soundness of the end-active variant, which is shown by translating derivations back into derivations in the corresponding sequent calculus.

Similar to the argument for *levelled derivations* in [Masini 1992, p. 241, Prop. 2], the following Lemmata show that the propositional rules can be restricted to be end-active. The first step is to show invertibility of the general forms of the propositional rules. Since all our calculi include the contraction rule we show this in a slightly more general form.

Definition 3.16. If $\Gamma_1 \Rightarrow \Delta_1 \parallel \dots \parallel \Gamma_n \Rightarrow \Delta_n$ is a linear nested sequent, then the *level* of the occurrences of formulae in Γ_i, Δ_i is i .

In all the results stated in this section, we will assume that $\mathcal{A} \subseteq \{N, M, C\}$.

LEMMA 3.17 (ADMISSIBILITY OF WEAKENING). *The weakening rule W is depth-preserving admissible in $\text{LNS}_{E, \mathcal{A}}$, i.e., if there is a derivation \mathcal{D} of $\mathcal{S}\{\Gamma \Rightarrow \Delta\}$ with depth at most n , then there is a derivation \mathcal{D}' of $\mathcal{S}\{\Gamma, \Sigma \Rightarrow \Pi, \Delta\}$ with depth at most n . Moreover, if the level of the active components of every rule application in \mathcal{D} is at least k , then the same holds for \mathcal{D}' .*

PROOF. As usual by induction on the depth of the derivation: applications of weakening are permuted upwards over every rule until they are absorbed by the initial sequents. Since this does not change the structure of the derivation and in particular does not introduce any new rule applications, the depth of the derivation and the minimal level of the active components of the rule applications is preserved. \square

LEMMA 3.18 (MULTI-INVERTIBILITY OF THE PROPOSITIONAL RULES). *The non-end-active versions of the propositional rules are m-invertible in $\text{LNS}_{E, \mathcal{A}}$, i.e., for every $n \geq 1$ we have:*

- (1) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, (\neg A)^n \Rightarrow \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow A^{n+k}, \Delta\}$ for some $k \geq 0$.*
- (2) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow (\neg A)^n, \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, A^{n+k} \Rightarrow \Delta\}$ for some $k \geq 0$.*
- (3) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, (A \rightarrow B)^n \Rightarrow \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, B^{n+k} \Rightarrow \Delta\}$ and $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow A^{n+\ell}, \Delta\}$ for some $k, \ell \geq 0$.*
- (4) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow (A \rightarrow B)^n, \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, A^{n+k} \Rightarrow B^{n+\ell}, \Delta\}$ for some $k, \ell \geq 0$.*
- (5) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, (A \vee B)^n \Rightarrow \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, A^{n+k} \Rightarrow \Delta\}$ and $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, B^{n+\ell} \Rightarrow \Delta\}$ for some $k, \ell \geq 0$.*
- (6) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow (A \vee B)^n, \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow A^{n+k}, B^{n+\ell}, \Delta\}$ for some $k, \ell \geq 0$.*
- (7) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, (A \wedge B)^n \Rightarrow \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma, A^{n+k}, B^{n+\ell} \Rightarrow \Delta\}$ for some $k, \ell \geq 0$.*
- (8) *If $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow (A \wedge B)^n, \Delta\}$, then also $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow A^{n+k}, \Delta\}$ and $\vdash_{\text{LNS}_{E, \mathcal{A}}} \mathcal{S}\{\Gamma \Rightarrow B^{n+\ell}, \Delta\}$ for some $k, \ell \geq 0$.*

Moreover, both the depth of the derivation and the minimal level of the active components of rule applications are preserved.

PROOF. By induction on the depth of the derivation, distinguishing cases according to the last applied rule. E.g., for the rule \rightarrow_R we have: if $\mathcal{S}\{\Gamma \Rightarrow (A \rightarrow B)^n, \Delta\}$ is an initial sequent or the conclusion of one of the rules \perp_L or \top_R , then so is the linear nested sequent $\mathcal{S}\{\Gamma, A^{n+k} \Rightarrow B^{n+\ell}, \Delta\}$ for any $k, \ell \geq 0$. If the last applied rule was not a contraction rule, we apply the induction hypothesis to its premiss(es), followed by the same rule. E.g., if the last applied rule was the rule C , and the

component containing $(A \rightarrow B)^n$ is the penultimate one, we have a derivation ending in

$$\frac{\begin{array}{c} \vdots \\ \mathcal{G} // \Gamma' \Rightarrow (A \rightarrow B)^n, \Delta //_{\mathbf{e}} (\Sigma, C \Rightarrow \Pi; \Omega \Rightarrow \Theta) \end{array} \quad \begin{array}{c} \vdots \\ \mathcal{G} // \Gamma' \Rightarrow (A \rightarrow B)^n, \Delta // \Omega \Rightarrow C, \Theta \end{array}}{\mathcal{G} // \Gamma', \Box C \Rightarrow (A \rightarrow B)^n \Delta //_{\mathbf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \mathbf{C}$$

Using the induction hypothesis, for some i, j, k, ℓ we obtain derivations of $\mathcal{G} // \Gamma', A^{n+i} \Rightarrow B^{n+j}, \Delta //_{\mathbf{e}} (\Sigma, C \Rightarrow \Pi; \Omega \Rightarrow \Theta)$ and $\mathcal{G} // \Gamma', A^{n+k} \Rightarrow B^{n+\ell}, \Delta // \Omega \Rightarrow C, \Theta$ and admissibility of weakening (Lemma 3.17) followed by an application of \mathbf{C} yields the desired $\mathcal{G} // \Gamma', \Box C, A^{n+\max\{i,k\}} \Rightarrow B^{n+\max\{j,\ell\}}, \Delta //_{\mathbf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)$. Finally, if the last applied rule was the contraction rule, we simply apply the induction hypothesis to its premiss. E.g., if the contracted formula is $A \rightarrow B$ and we have a derivation ending in

$$\frac{S\{\Gamma \Rightarrow (A \rightarrow B)^{n+1}, \Delta\}}{S\{\Gamma \Rightarrow (A \rightarrow B)^n, \Delta\}} \text{ICR}$$

we use the induction hypothesis to obtain $S\{\Gamma, A^{n+1+k} \Rightarrow B^{n+1+\ell}, \Delta\}$ for some $k, \ell \geq 0$. \square

Of course, setting $n = 1$ in the statement of the previous lemma and (possibly) applying a number of contractions to the result recovers standard invertibility of the propositional rules, albeit not the depth-preserving version.

Using this we first obtain soundness of the full calculus with respect to the end-active variant.

LEMMA 3.19. *If a linear nested sequent $\Gamma \Rightarrow \Delta$ is derivable in $\text{LNS}_{\mathbf{E}\mathcal{A}}$, then it is derivable in the end-active variant of $\text{LNS}_{\mathbf{E}\mathcal{A}}$.*

PROOF. Due to the nature of the modal rules it is clear that in a derivation only applications of the propositional rules and contraction can violate the end-activeness condition. We then successively transform a derivation of $\Gamma \Rightarrow \Delta$ into an end-active derivation as follows. Take the bottom-most block of modal rules such that there is an application of a propositional rule or contraction above it with level of the active component smaller than the maximal level of the active components in the modal block. Since the modal rules only apply to formulae in the last component, all such applications of propositional rules introduce a propositional connective which in the conclusion of the modal block is not under a modality. Using multi-invertibility of the propositional connectives (Lemma 3.18) we replace every such formula in the conclusion of the modal block by its constituents, possibly with multiplicity more than one. E.g, if the conclusion of the modal block has the form

$$\frac{\begin{array}{c} \vdots \\ \mathcal{G} // \Gamma \Rightarrow A \rightarrow B, \Delta // \mathcal{H} // \Sigma \Rightarrow \Pi //_{\mathbf{e}} (\Rightarrow C; C \Rightarrow) \end{array}}{\mathcal{G} // \Gamma \Rightarrow A \rightarrow B, \Delta // \mathcal{H} // \Sigma \Rightarrow \Box C, \Pi} \Box_R^{\mathbf{e}}$$

with the formula $A \rightarrow B$ introduced above the modal block, using m-invertibility we obtain

$$\frac{\begin{array}{c} \vdots \\ \mathcal{D} \\ \mathcal{G} // \Gamma, A^{1+k} \Rightarrow B^{1+\ell}, \Delta // \mathcal{H} // \Sigma \Rightarrow \Pi //_{\mathbf{e}} (\Rightarrow C; C \Rightarrow) \end{array}}{\mathcal{G} // \Gamma, A^{1+k} \Rightarrow B^{1+\ell}, \Delta // \mathcal{H} // \Sigma \Rightarrow \Box C, \Pi}$$

Then we delete every application of the contraction rule with active component of level smaller than the maximal level of the active components in the modal block from the derivation, possibly using Lemma 3.17 to ensure that the contexts in two-premiss rules are the same. From the proof of Lemma 3.18 it can be seen that afterwards the minimal level of the active components in rule applications in the derivation up to the conclusion of the modal block is at least the maximal level of the active components in the modal block itself. Finally, we use end-active applications of contraction to remove unwanted duplicates followed by end-active applications of the propositional rules to

reintroduce the propositional connectives in the right place, i.e., when the component containing the constituent formulae is the last one. Since the conclusion of the original derivation contained only a single component, this is always possible. \square

From this we obtain soundness of the full calculus by first translating derivations into derivations in the end-active variant, then into derivations in the corresponding sequent calculus:

THEOREM 3.20 (SOUNDNESS). *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\text{LNS}_{\mathcal{E}\mathcal{A}}$ for $\mathcal{A} \subseteq \{\mathbf{N}, \mathbf{M}, \mathbf{C}\}$, then it is derivable in the corresponding sequent calculus.*

PROOF. From the previous lemma we obtain that if a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\text{LNS}_{\mathcal{E}\mathcal{A}}$, then it is derivable in the end-active variant of $\text{LNS}_{\mathcal{E}\mathcal{A}}$. A derivation of the latter form then is translated into a $\mathcal{G}_{\mathcal{E}\mathcal{A}}$ derivation, discarding everything apart from the last component of the linear nested sequents, and translating blocks of modal rules into the corresponding modal sequent rules. E.g., a block consisting of an application of \Box_L^e followed by n applications of \mathbf{C} and an application of \Box_R^e is translated into an application of the rule (\mathbf{En}) . In the monotone case we use the fact that the rule \mathbf{M} permutes down over the rule \mathbf{C} , i.e., a modal block

$$\begin{array}{c}
 \frac{\mathcal{G} // \Gamma \Rightarrow \Delta // A_1, \dots, A_n \Rightarrow B \quad \overline{\mathcal{G} // \Gamma, A_n \Rightarrow \Delta // B, \perp \Rightarrow A_n}}{\mathcal{G} // \Gamma, A_n \Rightarrow \Delta //_{\mathbf{e}}(A_1, \dots, A_{n-1} \Rightarrow B; B, \perp \Rightarrow)} \frac{\perp_L}{\Box_L^e} \\
 \vdots \\
 \frac{\mathcal{G} // \Gamma, \Box A_{k+1}, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(A_1, \dots, A_k \Rightarrow B; B, \perp \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_{k+1}, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(A_1, \dots, A_k \Rightarrow B; B \Rightarrow)} \mathbf{M} \\
 \vdots \\
 \frac{\mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(A_1 \Rightarrow B; B \Rightarrow) \quad \mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta // B \Rightarrow A_1}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(\Rightarrow B; B \Rightarrow)} \mathbf{C} \\
 \frac{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(\Rightarrow B; B \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \Box_R^e
 \end{array}$$

is first turned into the following block by permuting the rule \mathbf{M} downwards and closing the derivations of the superfluous premisses using the \perp_L rule:

$$\begin{array}{c}
 \frac{\mathcal{G} // \Gamma \Rightarrow \Delta // A_1, \dots, A_n \Rightarrow B \quad \overline{\mathcal{G} // \Gamma, A_n \Rightarrow \Delta // B, \perp \Rightarrow A_n}}{\mathcal{G} // \Gamma, A_n \Rightarrow \Delta //_{\mathbf{e}}(A_1, \dots, A_{n-1} \Rightarrow B; B, \perp \Rightarrow)} \frac{\perp_L}{\Box_L^e} \\
 \vdots \\
 \frac{\mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(A_1 \Rightarrow B; B, \perp \Rightarrow) \quad \overline{\mathcal{G} // \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta // B, \perp \Rightarrow A_1}}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(\Rightarrow B; B, \perp \Rightarrow)} \frac{\perp_L}{\mathbf{C}} \\
 \frac{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(\Rightarrow B; B, \perp \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(\Rightarrow B; B \Rightarrow)} \mathbf{M} \\
 \frac{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\mathbf{e}}(\Rightarrow B; B \Rightarrow)}{\mathcal{G} // \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \Box_R^e
 \end{array}$$

The resulting modal block then is translated into an application of the rule (\mathbf{Mn}) . The propositional rules only work on the last component, never inside the nesting $//_{\mathbf{e}}$ and are translated easily by the corresponding sequent rules. \square

Note that due to the following lemma for the logics of the non-normal cube we could have avoided the complications arising from including the contraction rules in the calculi. However, in view of the calculi in later sections we chose the given more general method for proving soundness.

LEMMA 3.21 (ADMISSIBILITY OF CONTRACTION). *Contraction is admissible in the calculus $\text{LNS}_{\mathcal{E}\mathcal{A}}$ without contraction, that is, if there is a derivation \mathcal{D} of $S\{\Gamma, A, A \Rightarrow \Delta\}$ (resp. $S\{\Gamma \Rightarrow \Delta, A, A\}$) not using \mathbf{ICL} , \mathbf{ICR} , then there is a derivation \mathcal{D}' of $S\{\Gamma, A \Rightarrow \Delta\}$ (resp. $S\{\Gamma \Rightarrow \Delta, A\}$) not using \mathbf{ICL} , \mathbf{ICR} .*

PROOF. The admissibility of contraction for propositional connectives derives standardly from the invertibility of propositional rules (Lemma 3.18). The cases where A is not principal are also standard.

Suppose that $\mathcal{S}\{\Gamma, \Box A, \Box A \Rightarrow \Delta\}$ has a proof of the shape

$$\frac{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\Theta} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta) \quad \mathcal{G} // \Gamma, \Box A \Rightarrow \Delta // \Omega \Rightarrow A, \Theta}{\mathcal{G} // \Gamma, \Box A, \Box A \Rightarrow \Delta //_{\Theta} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \text{C}$$

Note that the $\Box A$ in the penultimate component of the conclusion of π_2 will be necessarily weakened, since no logical rules can act on it. In the proof π_1 , either $\Box A$ is never active, in which case it can be weakened, or it is active via one of the rules C or \Box_L^e . Let's consider the first case:

$$\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\Theta} (\Sigma, A, A \Rightarrow \Pi; \Omega \Rightarrow \Theta) \quad \mathcal{G} // \Gamma \Rightarrow \Delta // \Omega \Rightarrow A, \Theta}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\Theta} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \text{C}$$

Observe that the modal block in π_1' will eventually end by producing leaves of the form $\mathcal{G} // \Gamma' \Rightarrow \Delta' // \Sigma', A, A \Rightarrow \Pi'$ and $\Omega' \Rightarrow \Theta'$. By induction hypothesis, for every proof for a sequent of the first form there is a proof of $\mathcal{G} // \Gamma' \Rightarrow \Delta' // \Sigma', A \Rightarrow \Pi'$. Hence, starting from such leaves and applying the same sequence of rules as in the modal block of π_1' , we have a proof π of $\mathcal{G} // \Gamma \Rightarrow \Delta //_{\Theta} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)$. Thus

$$\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\Theta} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta) \quad \mathcal{G} // \Gamma \Rightarrow \Delta // \Omega \Rightarrow A, \Theta}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\Theta} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \text{C}$$

The other cases are similar and simpler. \square

It is worth noting that modular calculi for the logics in the non-normal cube were also given in [Gilbert and Maffezioli 2015] using the framework of labelled sequents. The calculi presented there are very much semantically motivated and are based on a translation of non-normal modal logics into normal modal logics. The complexity of the resulting semantic conditions then is captured using *systems of rules* [Negri 2016].

4. STRUCTURAL VARIANTS AND THE MODAL TESSERACT

The systems for the non-normal logics introduced in the last section make use of different *logical* rules, but sometimes it is preferable to change logics only by modifying the *structural* rules of the system. In particular, for sequent systems varying structural rules instead of logical rules often results in higher modularity, since cut elimination proofs are usually less affected by additional structural rules. This has also been called *Došen's Principle* in [Wansing 2002]. We will now apply this idea to obtain modular calculi for a number of extensions of monotone modal logic \mathbf{M} (see also [Hansen 2003] for a semantic treatment not only of these logics). In order to do so, we first simplify the calculus for monotone modal logic. As the avid reader might have noticed, there is quite a lot of redundancy in this calculus. In particular, after applying the rule \mathbf{M} , the second premiss of the following applications of C or \Box_L^e become trivially derivable. Hence for the present purpose we might as well omit these premisses and the corresponding component of the nesting operator, replacing the binary operator $//_{\Theta}$ with the unary operator $//_{\mathbf{m}}$. Linear nested sequents for monotone modal logics then are given by:

$$\text{LNS}_{\mathbf{m}} ::= \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta //_{\mathbf{m}} \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta // \text{LNS}_{\mathbf{m}}$$

The rules \Box_R^e and \Box_L^e in the monotone setting then are simplified to the rules $\Box_R^{\mathbf{m}}$ and $\Box_L^{\mathbf{m}}$ of Fig. 8, which now only need to carry information about one direction of the premisses. The additional rules for the axioms C and \mathbf{M} (shown in the same figure) now are given in their structural variants, permitting to switch from the “finished rule” marker $//$ to the “unfinished rule” marker $//_{\mathbf{m}}$ and back. Obviously, adding both rules \mathbf{N} and C collapses both nesting operators into one, and essentially brings us to the linear nested sequent calculus for modal logic \mathbf{K} from Fig. 2, as should be the case since \mathbf{K} is precisely the logic \mathbf{MNC} . Finally, observe that applying rule C allows propositional rules to be applied between modal phases.

$$\begin{array}{c}
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\mathbf{m}} \Rightarrow B}{\mathcal{G} // \Gamma \Rightarrow \Box B, \Delta} \Box_R^{\mathbf{m}} \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta // \Sigma, A \Rightarrow \Pi}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\mathbf{m}} \Sigma \Rightarrow \Pi} \Box_L^{\mathbf{m}} \quad \frac{\mathcal{G} //_{\mathbf{m}} \Gamma \Rightarrow \Delta}{\mathcal{G} // \Gamma \Rightarrow \Delta} \mathbf{C} \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta}{\mathcal{G} //_{\mathbf{m}} \Gamma \Rightarrow \Delta} \mathbf{N} \\
\text{LNS}_{\mathbf{M}\mathcal{A}} \quad \{ \Box_R^{\mathbf{m}}, \Box_L^{\mathbf{m}} \} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\mathbf{C}, \mathbf{N}\}
\end{array}$$

Fig. 8. The structural variants of the linear nested systems for monotone modal logics

The main benefit of capturing the axioms **C** and **N** by structural rules instead of logical rules is that it is now possible to give calculi for further extensions in a uniform way, independent of normality or non-normality of the base logic. The further axioms we are going to consider are (using the terminology of [Hansen 2003]):

$$\mathbf{P} \neg\Box\bot \quad \mathbf{D} \neg(\Box A \wedge \Box\neg A) \quad \mathbf{T} \Box A \rightarrow A \quad \mathbf{4} \Box A \rightarrow \Box\Box A \quad \mathbf{5} \Box A \vee \Box\neg\Box A$$

Note that we included both the two axioms **P** and **D** which are usually taken to be two different formulations of the axiom for seriality. This is due to the fact that in the non-normal setting the two formulations are not equivalent: while **P** is derivable from **D**, the opposite does not hold in logics not validating the axiom **C**. The reason for why we here only consider extensions of monotone modal logic with these axioms instead of extensions of classical modal logic **E** is that obtaining cut-free sequent calculi for many of these extensions seems to be problematic, see e.g. [Indrzejczak 2011].

Definition 4.1 (Sequent calculi). The sequent rules for extensions of monotone modal logic with axioms from $\{\mathbf{P}, \mathbf{D}, \mathbf{T}, \mathbf{4}, \mathbf{5}\}$ are given in Fig. 9. Let $\mathcal{A} \subseteq \{\mathbf{N}, \mathbf{P}, \mathbf{D}, \mathbf{T}, \mathbf{4}, \mathbf{5}\}$. The sequent system $\mathbf{G}_{\mathbf{M}\mathcal{A}}$ contains the standard propositional and structural rules of **G** (see Def. 3.6) as well as the following modal rules:

- $\{\mathbf{M}\} \cup \mathcal{A}$
- **D4** if $\{\mathbf{D}, \mathbf{4}\} \subseteq \mathcal{A}$
- **D5** if $\{\mathbf{D}, \mathbf{5}\} \subseteq \mathcal{A}$.

The sequent system $\mathbf{G}_{\mathbf{MC}\mathcal{A}}$ contains the rules standard propositional and structural rules together with the additional rules

- $\{\mathbf{C}\} \cup \mathcal{A}$
- **CD** if $\mathbf{P} \in \mathcal{A}$ or $\mathbf{D} \in \mathcal{A}$
- **C4** if $\mathbf{4} \in \mathcal{A}$
- **CD4** if $\{\mathbf{P}, \mathbf{4}\} \subseteq \mathcal{A}$ or $\{\mathbf{D}, \mathbf{4}\} \subseteq \mathcal{A}$
- **K4** if $\{\mathbf{N}, \mathbf{4}\} \subseteq \mathcal{A}$
- **K45** if $\{\mathbf{N}, \mathbf{4}, \mathbf{5}\} \subseteq \mathcal{A}$
- **KD45** if $\{\mathbf{N}, \mathbf{P}, \mathbf{4}, \mathbf{5}\} \subseteq \mathcal{A}$ or $\{\mathbf{N}, \mathbf{D}, \mathbf{4}, \mathbf{5}\} \subseteq \mathcal{A}$

The additional sequent rules stipulated in the above definition are required for cut elimination. Decomposing the rules yields the linear nested sequent rules and rule sets $\text{LNS}_{\mathbf{M}\mathcal{A}}$ given in Fig. 10. Note in particular that we do not need to include additional rules, and hence the calculi are completely modular. As for normal modal logics, extensions of **M** including the axiom **5** are not as well behaved as those without it. We first consider logics not including **5**. Most of the following results can be found in the literature.

PROPOSITION 4.2. *For $\mathcal{A} \subseteq \{\mathbf{N}, \mathbf{C}, \mathbf{P}, \mathbf{D}, \mathbf{4}\}$ the sequent calculus $\mathbf{G}_{\mathbf{M}\mathcal{A}}$ is sound and complete for the logic $\mathbf{M}\mathcal{A}$.*

PROOF. For the extensions of normal modal logic $\mathbf{K} = \mathbf{MNC}$, see e.g. [Wansing 2002]. For the logic **MCT** the result is shown in [Ohnishi and Matsumoto 1957], for the extensions of **M** with axioms from $\{\mathbf{N}, \mathbf{C}\}$ see [Lavendhomme and Lucas 2000]. The result for the logics **MP** and **MCP** = **MCD** can be found in [Orlandelli 2014]. The majority of the results for the non-normal logics are due to [Indrzejczak 2005], namely the calculi for all extensions of **M** with axioms from $\{\mathbf{N}, \mathbf{D}, \mathbf{T}, \mathbf{4}\}$.

$$\begin{array}{c}
\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} \text{ M} \quad \frac{\Rightarrow A}{\Rightarrow \Box A} \text{ N} \quad \frac{A \Rightarrow}{\Box A \Rightarrow} \text{ P} \quad \frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow} \text{ D} \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \text{ T} \\
\\
\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \frac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B} 5 \quad \frac{A, \Box B \Rightarrow}{\Box A, \Box B \Rightarrow} \text{ D4} \quad \frac{A \Rightarrow \Box B}{\Box A \Rightarrow \Box B} \text{ D5} \\
\\
\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{ C} \quad \frac{\Gamma \Rightarrow}{\Box \Gamma \Rightarrow} \text{ CD} \quad \frac{\Box \Gamma, \Sigma \Rightarrow B}{\Box \Gamma, \Box \Sigma \Rightarrow \Box B} \text{ C4} \quad \frac{\Box \Gamma, \Sigma \Rightarrow}{\Box \Gamma, \Box \Sigma \Rightarrow} \text{ CD4} \\
(|\Gamma| \geq 1) \quad (|\Gamma, \Sigma| \geq 1) \\
\\
\frac{\Box \Gamma, \Sigma \Rightarrow A}{\Box \Gamma, \Box \Sigma \Rightarrow \Box A} \text{ K4} \quad \frac{\Box \Gamma, \Sigma \Rightarrow A, \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box A, \Box \Delta} \text{ K45} \quad \frac{\Box \Gamma, \Sigma \Rightarrow \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box \Delta} \text{ KD45}
\end{array}$$

Fig. 9. Sequent rules for extensions of monotonic logics. We slightly abuse notation and write the same letters for axioms and the corresponding rules.

$$\begin{array}{c}
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\text{m}} \Rightarrow}{\mathcal{G} // \Gamma \Rightarrow \Delta} \text{ P} \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\text{m}} A \Rightarrow}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta} \text{ D} \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\text{m}} \Sigma \Rightarrow \Pi}{\mathcal{G} // \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ T} \\
\\
\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\text{m}} \Sigma, \Box A \Rightarrow \Pi}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\text{m}} \Sigma \Rightarrow \Pi} 4 \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\text{m}} \Sigma \Rightarrow \Pi, \Box A}{\mathcal{G} // \Gamma \Rightarrow \Delta, \Box A //_{\text{m}} \Sigma \Rightarrow \Pi} 5 \\
\\
\text{LNS}_{\text{M}\mathcal{A}} \quad \{\Box_R^{\text{m}}, \Box_L^{\text{m}}\} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\text{C, N, P, D, 4, 5}\}
\end{array}$$

Fig. 10. Linear nested sequent rules for extensions of monotonic modal logics

The remaining calculi for the logics MP, MP4, MNP, MNP4, MC4, MCP4 = MCD4 and MT4 are easily constructed using methods similar to the ones in [Lellmann and Pattinson 2013; Lellmann 2013]. The cut elimination proof for these calculi essentially is an extension of the cut elimination proof given in [Indrzejczak 2005]. Since it is not central to the topic of this paper we relegate it to Appendix A. \square

This gives modular systems for every combination of C, N, P, D, T, 4.

THEOREM 4.3 (SOUNDNESS AND COMPLETENESS). *Let \mathcal{A} be a subset of $\{\text{C, N, P, D, T, 4}\}$. Then the linear nested sequent calculus $\text{LNS}_{\text{M}\mathcal{A}}$ is sound and complete for the logic $\text{M}\mathcal{A}$.*

PROOF. We first show completeness by simulating sequent derivations in the last component. Here we only show how to simulate the modal rules. First, the rules (Mn) for monotone logics including the axiom C are simulated by

$$\frac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta} \text{ (Mn)} \quad \rightsquigarrow \quad \frac{\frac{\Gamma \Rightarrow \Delta // A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_n \Rightarrow \Delta //_{\text{m}} A_1, \dots, A_{n-1} \Rightarrow B} \Box_L^{\text{m}} \quad \vdots \quad \frac{\Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta //_{\text{m}} A_1 \Rightarrow B}{\Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta //_{\text{m}} A_1 \Rightarrow B} \text{ C} \quad \Box_L^{\text{m}}}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta //_{\text{m}} \Rightarrow B} \Box_R^{\text{m}}$$

The case for $n = 1$ gives the simulation of the rule M. The necessitation rule N is simulated by:

$$\frac{\Rightarrow B}{\Gamma \Rightarrow \Box B, \Delta} N \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow \Delta // \Rightarrow B}{\Gamma \Rightarrow \Delta // \Box B \Rightarrow B} N \quad \frac{}{\Gamma \Rightarrow \Box B, \Delta} \Box_R^m$$

The simulations of the rules from Fig. 10 then are (omitting the general context \mathcal{G}):

$$\begin{array}{ccc} \frac{A \Rightarrow}{\Box A \Rightarrow} P & \rightsquigarrow & \frac{\Rightarrow // A \Rightarrow}{\Box A \Rightarrow // \Box A \Rightarrow} \Box_L^m \quad \frac{}{\Box A \Rightarrow} P \\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} T & \rightsquigarrow & \frac{\Rightarrow // \Gamma, A \Rightarrow \Delta}{\Box A \Rightarrow // \Box \Gamma \Rightarrow \Delta} \Box_L^m \quad \frac{}{\Gamma, \Box A \Rightarrow \Delta} T \\ \frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow} D & \rightsquigarrow & \frac{\Rightarrow // A, B \Rightarrow}{\Box A \Rightarrow // \Box B \Rightarrow} \Box_L^m \quad \frac{}{\Box A, \Box B \Rightarrow} D \\ \frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow} D4 & \rightsquigarrow & \frac{\Gamma \Rightarrow \Delta // A, \Box B \Rightarrow}{\Gamma, \Box B \Rightarrow \Delta // \Box A \Rightarrow} 4 \quad \frac{}{\Gamma, \Box A, \Box B \Rightarrow \Delta} D \end{array}$$

In the presence of C we use the rule C to move additional formulae on the left hand side. E.g., for the system containing the axioms C , D and 4 we would have:

$$\frac{\Box B, A_1, A_2 \Rightarrow}{\Box B, \Box A_1, \Box A_2 \Rightarrow} CD4 \quad \rightsquigarrow \quad \frac{\Rightarrow // \Box B, A_1, A_2 \Rightarrow}{\Box A_2 \Rightarrow // \Box \Box B, A_1 \Rightarrow} \Box_L^m \quad \frac{}{\Box A_2 \Rightarrow // \Box B, A_1 \Rightarrow} C \quad \frac{}{\Box B, \Box A_2 \Rightarrow // \Box A_1 \Rightarrow} 4 \quad \frac{}{\Box B, \Box A_1, \Box A_2 \Rightarrow} D$$

If the logic contains P but not D , the application of the rule D above is replaced by an application of P followed by applications of \Box_L^m and C . The cases of the remaining rules C , CD , $C4$, $K4$ are analogous.

To show soundness we first observe that applications of the propositional rules in the last component can be permuted above blocks of applications between the rules C and \Box_L^m , i.e., above blocks of rule applications where the last nesting is $//_m$. This is due to the fact that only modal rules can be applied inside the nesting $//_m$, no modal rule creates a new nesting after a $//_m$ nesting, and every modal rule keeps all the formulae occurring under the nesting $//_m$ in the conclusion at the same place. Hence we may assume that in a derivation in $LNS_{M, \mathcal{A}}$ all the modal rules occur in a block. Then, analogously to Lemma 3.19 of the previous section, and using the formulations of Lemma 3.17 and 3.18 for the present calculi (the proofs of which are completely analogous), we convert the derivation into a derivation where all the applications of rules are end-active. Such a derivation then is converted into a sequent derivation. In particular, every modal block then can be translated into one or more modal rules in the sequent system: whenever we have a block

$$\frac{\mathcal{G} // \Gamma \Rightarrow \Delta // \Sigma \Rightarrow \Pi}{\mathcal{G} // \Gamma, \Sigma' \Rightarrow \Delta, \Pi'}$$

consisting only of modal rules, then the sequent rule

$$\frac{\Sigma \Rightarrow \Pi}{\Gamma, \Sigma' \Rightarrow \Delta, \Pi'}$$

is derivable in the corresponding sequent system. The transformations are essentially the backwards directions of the transformations given above. \square

Thus we obtain modular nested sequent calculi for all the logics in what could be called the *modal tesseract* (Fig. 11), hence repairing the bridge between non-normal and normal modal logics. Note

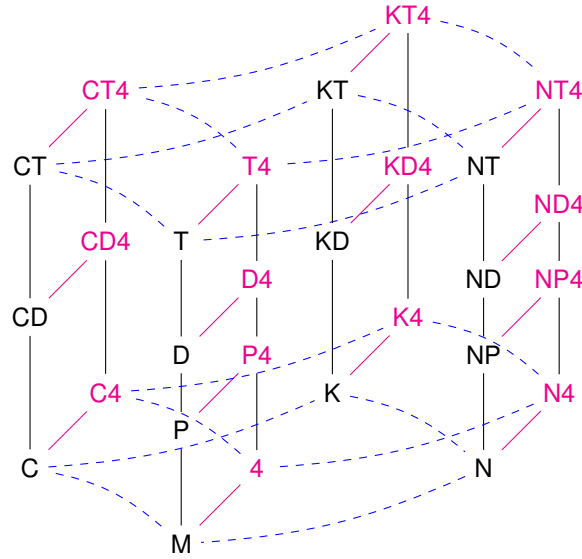


Fig. 11. The modal tesseract

that the modal tesseract includes one side of the standard (normal) modal cube, see e.g. [Brünnler 2009].

For logics including the axiom 5 the situation is a bit more complicated, since not all of these (in particular K5 and S5) have cut-free sequent calculi. However, while in this case we do not obtain full modularity, we still obtain calculi for a number of logics. The number of logics we need to consider in this case is greatly reduced by the following simple observation.

LEMMA 4.4. *The axiom N is derivable in any extension of M including an axiom of the form $\Box A_1 \vee \dots \vee \Box A_n$. In particular, the axiom N is derivable in M5.*

PROOF. Using the fact that $A_i \rightarrow \top$ is a tautology for every A_i and monotonicity. \square

Hence the lattice of extensions of M5 with axioms from $\{N, C, P, D, T, 4\}$ collapses to the 12 logics shown in Fig. 12 (the *house of M5* – not all corners of it are safe, i.e., cut-free, though). In particular, the extensions of MC5 are the same as the extensions of normal modal logic K5. Again, most of the following results are found in the literature.

PROPOSITION 4.5. *Let \mathcal{L} be one of the logics*

$$\{M5, MP5, M45, MP45, MD45, K45, KD45\}$$

Then $G_{\mathcal{L}}$ is sound and complete for \mathcal{L} .

PROOF. For the logics K45 and KD45 this was shown in [Shvarts 1989], but note that the calculi for K45 and KD45 considered here are slight variations of the ones in *op. cit.* In particular, there also the rule

$$\frac{\Box \Gamma, \Sigma \Rightarrow \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box \Delta}$$

with nonempty Δ is included in the rule set for K45. This rule can be derived in the calculus considered here using the rule K45 together with a cut on the derivable sequent $\Box \Box A \Rightarrow \Box A$. Equivalence of the cut-free systems then follows from cut elimination for G_{K45} (using e.g. [Lellmann 2013, Thm. 2.3.16]).

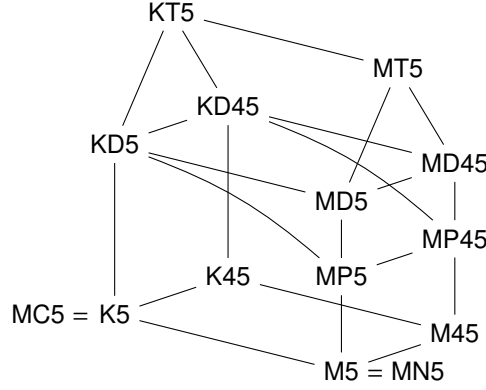


Fig. 12. The extensions of modal logic M5

In the non-normal case, for the logics M5, M45 and MD45 the result can be found in [Indrzejczak 2005]. The result for the remaining two logics MP5 and MP45 follows similarly to the results of Prop. 4.2 from the cut elimination proof in Appendix A. \square

For all the other cases there are counterexamples to cut elimination. In particular, for the non-normal logic MD5 this is given e.g. by the formula $\Box p \rightarrow \Box \Diamond p$, which is derivable using the instance $\Box p \rightarrow \Diamond p$ of axiom D and the instance $\Diamond p \rightarrow \Box \Diamond p$ of axiom 5, but not cut-free derivable in \mathbf{G}_{MD5} . Interestingly, this formula is not a theorem of MP5, and hence not a counterexample to cut elimination for \mathbf{G}_{MP5} .¹

THEOREM 4.6. *Let \mathcal{L} be one of the logics*

$$\{\mathbf{M5}, \mathbf{MP5}, \mathbf{M45}, \mathbf{MP45}, \mathbf{MD45}, \mathbf{K45}, \mathbf{KD45}\}$$

Then $\mathbf{LNS}_{\mathcal{L}}$ is sound and complete for \mathcal{L} .

PROOF. Analogous to the proof of Thm. 4.3. The missing transformations from sequent rules into linear nested sequent derivations are:

$$\frac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B} 5 \quad \rightsquigarrow \quad \frac{\frac{\Rightarrow // \Rightarrow A, \Box B}{\Rightarrow \Box B //_{\mathbf{m}} \Rightarrow A} 5}{\Rightarrow \Box A, \Box B} \Box_R^{\mathbf{m}} \quad \frac{A \Rightarrow \Box B}{\Box A \Rightarrow \Box B} \mathbf{D5} \quad \rightsquigarrow \quad \frac{\frac{\Gamma \Rightarrow \Delta // A \Rightarrow \Box B}{\Gamma \Rightarrow \Box B, \Delta //_{\mathbf{m}} A \Rightarrow} 5}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \mathbf{D}$$

The rules K45 and KD45 are transformed similar to the case of CD4.

The soundness proof is analogous to the one for the cases not involving the axiom 5. \square

5. RECONCILING SEQUENTS WITH NESTED SEQUENTS

We have presented a modular way of proposing several different modal systems. The beauty in this is that all systems share the same core, where modal rules can be plugged in and/or mixed together. For that, we refined sequent rules, exposing their behaviour locally. The price to pay for this modularity is, of course, efficiency, since there are more rules which could have been applied to derive a given sequent. In particular, the propositional rules could be applied in any component, giving rise to a great number of derivations which should be identified modulo bureaucracy. This alone could be taken care of by simply restricting the calculi to their end-active variants, so that the propositional rules are applied only in the last component. However, doing so would still leave open the possibility of mixing propositional and modal rules, e.g., applying (bottom-up) a rule \Box_{iR} followed by a propositional rule

¹For the reader familiar with neighbourhood semantics [Chellas 1980; Hansen 2003]: The $\mathbf{MP5}$ -model $(\{a, b\}, \eta, \sigma)$ with $\eta(a) = \eta(b) = \{\{a\}, \{b\}, \{a, b\}\}$ and $\llbracket p \rrbracket = \{a\}$ witnesses satisfiability of the negation of this formula.

$$\begin{array}{c}
\frac{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta \parallel^j \Sigma, A \Rightarrow \Pi}{\mathcal{G} \parallel^k \Gamma, \Box_i A \Rightarrow \Delta \parallel^j \Sigma \Rightarrow \Pi} \Box_{ijL} \quad \frac{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta \parallel^i \Rightarrow A}{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta, \Box_i A} \Box_{iR} \quad \frac{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta}{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta} \text{close} \\
\\
\frac{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta \parallel^j A \Rightarrow}{\mathcal{G} \parallel^k \Gamma, \Box_i A \Rightarrow \Delta} \mathbf{d}_{ij} \quad \frac{\mathcal{G} \parallel^k \Gamma, \Box_i A, A \Rightarrow \Delta}{\mathcal{G} \parallel^k \Gamma, \Box_i A \Rightarrow \Delta} \mathbf{t}_i \quad \frac{\mathcal{G} \parallel^k \Gamma \Rightarrow \Delta \parallel^j \Sigma, \Box_i A \Rightarrow \Pi}{\mathcal{G} \parallel^k \Gamma, \Box_i A \Rightarrow \Delta \parallel^j \Sigma \Rightarrow \Pi} \mathbf{4}_{ij}
\end{array}$$

Fig. 13. Modal rules for $\text{FLNS}_{(N, \leq, F)}$, where k, i, j are as in Figure 5. The propositional rules are the same as in Fig. 1, restricted to the last component.

in the last component, and then a rule \Box_{jiL} . This as well is a potential source of inefficiency when compared to the sequent framework, where we have blocks of propositional rules alternating with single modal rules.

In this section, we will show how auxiliary nesting operators can be used in order to guarantee a notion of “normal proofs” for LNS derivations that mimic the respective sequent ones, hence reducing the proof search space and optimizing proof search. As noted in the last sections all the systems in this work can be restricted to their end-active versions without losing completeness. Moreover, rules for the propositional connectives permute over box left rules. This allows modal rules to be restricted so that they occur *in a block*.

Definition 5.1. A LNS derivation is in *block form* if, whenever a modal rule occurs directly above a propositional rule, then that modal rule creates a new component.

Considering first the simply dependent normal multimodal logics of Section 3.1, in Fig. 13 we present $\text{FLNS}_{(N, \leq, F)}$, an end-active version for $\text{LNS}_{(N, \leq, F)}$ (Fig. 5) where all derivations are necessarily in block form: this is assured by an auxiliary nesting operator \parallel^i for each $i \in N$. This operator behaves much in the same way as the “unfinished rule marker” in the systems for non-normal modal logics. However, here we explicitly include the rule *close*, which intuitively marks a sequent rule as finished. This implies that, *modulo the order of application of \Box_{ijL} and \mathbf{d}_{ij} rules*, there is a 1-1 correspondence between derivations in the LNS system $\text{FLNS}_{(N, \leq, F)}$ and in the sequent system $\mathbf{G}_{(N, \leq, F)}$ (see Fig. 4). In this way, sequent rules can be seen as *macro rules* for linear nested rules. Since every $\text{FLNS}_{(N, \leq, F)}$ -derivation can be translated into a $\text{LNS}_{(N, \leq, F)}$ -derivation by replacing the nesting \parallel^i everywhere by \parallel^i and omitting every application of the rule *close* we immediately obtain soundness of the system $\text{FLNS}_{(N, \leq, F)}$. Completeness follows as mentioned above from permuting propositional rules below modal rules in derivations in the end-active variant of $\text{LNS}_{(N, \leq, F)}$.

Observe that the *normal* modal logics presented in this paper form a particular case of simply dependent multimodal logics (with N being a singleton). Hence all derivations in the correspondent $\text{FLNS}_{(N, \leq, F)}$ system will be in block form.

Example 5.2. The block form derivation for the normality axiom is as follows

$$\begin{array}{c}
\frac{\cdot \Rightarrow \cdot \parallel p \Rightarrow p, q}{\cdot \Rightarrow \cdot \parallel p \rightarrow q, p \Rightarrow q} \text{init} \quad \frac{\cdot \Rightarrow \cdot \parallel p, q \Rightarrow q}{\cdot \Rightarrow \cdot \parallel p \rightarrow q, p \Rightarrow q} \text{init} \\
\frac{\cdot \Rightarrow \cdot \parallel p \rightarrow q, p \Rightarrow q}{\cdot \Rightarrow \cdot \parallel p \rightarrow q, p \Rightarrow q} \text{close} \\
\frac{\cdot \Rightarrow \cdot \parallel p \rightarrow q, p \Rightarrow q}{\Box(p \rightarrow q), \Box p \Rightarrow \cdot \parallel \cdot \Rightarrow q} \Box_L \\
\frac{\Box(p \rightarrow q), \Box p \Rightarrow \cdot \parallel \cdot \Rightarrow q}{\Box(p \rightarrow q), \Box p \Rightarrow \Box q} \Box_R \\
\frac{\Box(p \rightarrow q), \Box p \Rightarrow \Box q}{\Rightarrow \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)} \rightarrow_R
\end{array}$$

All the systems presented for *non-normal* modal logics in the previous sections are end-active and in most systems the partial nesting operator already forces that all valid derivations are in block form. The exception are the systems containing the \mathbf{C} rule (Fig. 8). In fact, this rule allows a partial nesting to begin anywhere in the proof, not only after an application of a modal rule.

$$\frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\mathbf{m}} \Rightarrow B}{\mathcal{G} // \Gamma \Rightarrow \Box B, \Delta} \Box_R^{\mathbf{m}} \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta //_{\mathbf{m}} \Sigma, A \Rightarrow \Pi}{\mathcal{G} // \Gamma, \Box A \Rightarrow \Delta //_{\mathbf{m}} \Sigma \Rightarrow \Pi} \Box_L^{\mathbf{m}} \quad \frac{\mathcal{G} //_{\mathbf{m}} \Gamma \Rightarrow \Delta}{\mathcal{G} // \Gamma \Rightarrow \Delta} \mathbf{C} \quad \frac{\mathcal{G} // \Gamma \Rightarrow \Delta}{\mathcal{G} // \Gamma \Rightarrow \Delta} \text{close}$$

Fig. 14. System FLNS_{MC}.

An alternative set of rules for these systems is obtained by adding the nesting operator $//$ (so that the modal rules have two levels of partial processing), together with the **close** rule, that forces the modal block to end. We illustrate this in Fig. 14 for the system FLNS_{MC}. Again, soundness and completeness follow as above.

5.1. Blocked versus focused derivations

Since block form derivations entail a notion of normal forms in LNS, a natural question that arises is if it also entails a notion of focusing for nested modal systems, as discussed in [Chaudhuri et al. 2016] (see also [Andreoli 1992] for the basics of focusing).

It is interesting to note that, in passing from nesting to linear nesting, one loses the dualities of polarities of modal connectives. In fact, \Box_L and \Box_R rules are both non-invertible in LNS_K: while the left rule can be applied only after a right rule, for the right rule one has to choose a boxed formula to be processed. Hence there seems to be no natural way of polarizing the modal connectives presented in this paper.

Let's take a closer look at the sequent rule \mathbf{k}

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box \Gamma \Rightarrow \Box A, \Delta} \mathbf{k}$$

and its interpretations in nested and linear nested systems.

In the *nested* system proposed in [Chaudhuri et al. 2016] we can process all the existing right boxes in parallel (and this is invertible) and then move the left boxes one by one to all the nestings. A proof then proceeds by running all the possible traces in parallel, and finish whenever one or more of them succeed. Although considering the box left a positive connective leads to a complete proof system, it has an inherited negative behaviour that is ignored when adopting such polarization. Also, in the sequent rule \mathbf{k} , the box right should be chosen, which gives it a positive behaviour, also not taken into account in the focused system proposed in [Chaudhuri et al. 2016].

In contrast, this positive/negative behavior of box left and right rules is present in LNS_K. In fact, while \Box_R is not invertible, proposing a focused version of this rule would render the resulting system incomplete. And, as mentioned before, the \Box_L rule has the restriction that it can be applied only after a \Box_R rule is applied, hence it has a positive behaviour. But once the new component is created by the \Box_R rule, moving the left boxed formulae can be done in any order and this action is invertible, hence negative.

Thus, although modal blocks do not correspond to focusing, it produces a normal form that mimics the sequential behaviour and preserves the inherent positive/negative flavor of the box modality. Focusing, on the other side, produces normal forms that do not correspond to sequent derivations, hence the proof space is much bigger in (focused) nested systems than in (block form) linear nested systems.

6. LABELLED LINE SEQUENT SYSTEMS AND BIPOLES

One of the main advantages of the LNS calculi over the standard sequent calculi is that the modal operators have separate left and right rules, and that the number of principal formulae in the modal rules is bounded. This is the first step towards an encoding of the calculi in a universal framework such as Linear Logic, as done, e.g., in [Miller and Pimentel 2013] for classical and intuitionistic logic. Such an encoding is interesting, not just because it shows that the considered logics can be encoded in a particular framework and hence decision methods for that framework also yield decision methods for these logics, but also because this is the prerequisite for formally proving meta-theorems

such as cut elimination in the meta-level framework. However, to do so we will need to encode the structure of linear nested sequents as well. For this we will reformulate this structure in the language of *labelled sequents* (see, e.g., [Negri 2005; Negri and van Plato 2011]) using the correspondence between nested sequents and labelled tree sequents from [Goré and Ramanayake 2012].

In the following we present labelled sequent versions for LNS systems and show how to use such systems in order to generate bipole clauses in linear logic which adequately correspond to LNS modal rules.

6.1. Labelled systems

Let SV a countable infinite set of *state variables* (denoted by x, y, z, \dots), disjoint from the set of propositional variables. A *labelled formula* has the form $x : A$ where $x \in SV$ and A is a formula. If $\Gamma = \{A_1, \dots, A_n\}$ is a multiset of formulae, then $x : \Gamma$ denotes the multiset $\{x : A_1, \dots, x : A_n\}$ of labelled formulae. A (possibly empty) set of relation terms (*i.e.* terms of the form xRy , where $x, y \in SV$) is called a *relation set*. For a relation set \mathcal{R} , the *frame* $Fr(\mathcal{R})$ defined by \mathcal{R} is given by $(|\mathcal{R}|, \mathcal{R})$ where $|\mathcal{R}| = \{x \mid xRy \in \mathcal{R} \text{ or } yRx \in \mathcal{R} \text{ for some } y \in SV\}$. We say that a relation set \mathcal{R} is *treelike* if the frame defined by \mathcal{R} is a tree or \mathcal{R} is empty. A treelike relation set \mathcal{R} is called *linelike* if each node in \mathcal{R} has at most one child.

Definition 6.1. A *labelled line sequent* LLS is a labelled sequent $\mathcal{R}, X \Rightarrow Y$ where

- (1) \mathcal{R} is linelike;
- (2) if $\mathcal{R} = \emptyset$ then X has the form $x_0 : A_1, \dots, x_0 : A_n$ and Y has the form $x_0 : B_1, \dots, x_0 : B_m$ for some $x_0 \in SV$;
- (3) if $\mathcal{R} \neq \emptyset$ then every state variable x that occurs in either X or Y also occurs in \mathcal{R} .

A *labelled line sequent calculus* is a labelled sequent calculus whose initial sequents and inference rules are constructed from LLS.

Observe that, in LLS, if $xRy \in \mathcal{R}$ then $uRy \notin \mathcal{R}$ and $xRv \notin \mathcal{R}$ for any $u, v \in SV$ such that $u \neq x$ and $v \neq y$.

Since linear nested sequents form a particular case of nested sequents, the algorithm given in [Goré and Ramanayake 2012] can be used for generating LLS from LNS, and vice versa. However, one has to keep the linearity property invariant through inference rules. For example, the following labelled sequent rule

$$\frac{\mathcal{R}, xRy, X \Rightarrow Y, y : A}{\mathcal{R}, X, \Rightarrow Y, x : \Box A} \Box'_R$$

where y is fresh, is not adequate w.r.t. the system LNS_K , since there may exist $z \in |\mathcal{R}|$ such that $xRz \in \mathcal{R}$. That is, for labelled sequents in general, freshness alone is not enough for guaranteeing unicity of x in \mathcal{R} . And it does not seem to be trivial to assure this unicity by using logical rules without side conditions. To avoid this problem, we slightly modify the framework by restricting \mathcal{R} to singletons, that is, $\mathcal{R} = \{xRy\}$ will record only the two last components, in this case labelled by x and y , and by adding a base case $\mathcal{R} = \{x_0Rx_1\}$ for x_0, x_1 different state variables when there are no nested components. The rule for introducing \Box_R then is

$$\frac{xRy, X \Rightarrow Y, y : A}{zRx, X, \Rightarrow Y, x : \Box A} \Box_R$$

with y fresh. Note that this solution corresponds to recording the history of the proof search up to the last two steps similar to what is outlined in [Pfenning 2015], hence we are adopting an end-active version of LLS.

Definition 6.2. An *end-active* LLS is a singleton relation set \mathcal{R} together with a sequent $X \Rightarrow Y$ of labelled formulae, written $\mathcal{R}, X \Rightarrow Y$. The rules of an *end-active* LLS calculus are constructed from end-active labelled line sequents such that the active formulae in a premiss $xRy, X \Rightarrow Y$ are labelled with y and the labels of all active formulae in the conclusion are in its relation set.

$$\begin{array}{c}
\frac{}{zRx, X, x: p \Rightarrow x: p, Y} \text{init} \quad \frac{zRx, X, x: A, x: B \Rightarrow Y}{zRx, X, x: A \wedge B \Rightarrow Y} \wedge_L \quad \frac{zRx, X \Rightarrow x: A, Y \quad zRx, X \Rightarrow x: B, Y}{zRx, X \Rightarrow x: A \wedge B, Y} \wedge_R \\
\frac{}{zRx, X, x: \perp \Rightarrow Y} \perp_L \quad \frac{zRx, X \Rightarrow Y, x: A \quad zRx, X, x: B \Rightarrow Y}{zRx, X, x: A \rightarrow B \Rightarrow Y} \rightarrow_L \quad \frac{zRx, X, x: A \Rightarrow Y, x: B}{zRx, X \Rightarrow Y, x: A \rightarrow B} \rightarrow_R
\end{array}$$

Fig. 15. The end-active version of LLS_G. In rule init, p is atomic.

$$\frac{xRy, X, y: A \Rightarrow Y}{xRy, X, x: \Box A \Rightarrow Y} \mathbb{T}\mathbb{L}_x(\Box_L) \quad \frac{xRy, X \Rightarrow Y, y: A}{zRx, X \Rightarrow Y, x: \Box A} \mathbb{T}\mathbb{L}_x(\Box_R)$$

Fig. 16. The modal rules of LLS_K. The variable y in rule \Box_R is fresh.

From now on, we will use the end-active version of the propositional rules (see Fig. 15).

We will now show how to automatically generate LLS from LNS. This is possible since the key property of end-active LNS calculi is that rules can only move formulae “forward”, that is, either an active formula produces other formulae in the same component or in the next one.

Definition 6.3. For a state variable x , define the mapping $\mathbb{T}\mathbb{L}_x$ from LNS to end-active LLS as follows

$$\begin{aligned}
\mathbb{T}\mathbb{L}_{x_1}(\Gamma_1 \Rightarrow \Delta_1) &= x_0 R x_1, x_1 : \Gamma_1 \Rightarrow x_1 : \Delta_1 \\
\mathbb{T}\mathbb{L}_{x_n}(\Gamma_1 \Rightarrow \Delta_1 // \dots // \Gamma_n \Rightarrow \Delta_n) &= x_{n-1} R x_n, x_1 : \Gamma_1, \dots, x_n : \Gamma_n \Rightarrow x_1 : \Delta_1, \dots, x_n : \Delta_n \quad n > 0
\end{aligned}$$

with all state variables pairwise distinct.

It is straightforward to use $\mathbb{T}\mathbb{L}_x$ in order to construct a LLS inference rule from an inference rule of an end-active LNS calculus. The procedure, that can also be automatised, is the same as the one presented in [Goré and Ramanayake 2012], as we shall illustrate here.

Example 6.4. Consider the following application of the rule \Box_R of Fig. 2:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 // \dots // \Gamma_{n-1} \Rightarrow \Delta_{n-1} // \Gamma_n \Rightarrow \Delta_n // \Rightarrow A}{\Gamma_1 \Rightarrow \Delta_1 // \dots // \Gamma_{n-1} \Rightarrow \Delta_{n-1} // \Gamma_n \Rightarrow \Delta_n, \Box A} \Box_R$$

Applying $\mathbb{T}\mathbb{L}_x$ to the conclusion we obtain $x_{n-1} R x_n, X \Rightarrow Y, x_n : \Box A$, where $X = x_1 : \Gamma_1, \dots, x_n : \Gamma_n$ and $Y = x_1 : \Delta_1, \dots, x_n : \Delta_n$. Applying $\mathbb{T}\mathbb{L}_x$ to the premise we obtain $x_n R x_{n+1}, X \Rightarrow Y, x_{n+1} : A$. We thus obtain an application of the LLS rule

$$\frac{x_n R x_{n+1}, X \Rightarrow Y, x_{n+1} : A}{x_{n-1} R x_n, X \Rightarrow Y, x_n : \Box A} \mathbb{T}\mathbb{L}_x(\Box_R)$$

Fig. 16 presents the end-active labelled line sequent calculus LLS_K for K.

The following result follows readily by transforming derivations bottom-up.

THEOREM 6.5. $\Gamma \Rightarrow \Delta$ is provable in a certain end-active LNS calculus if and only if $\mathbb{T}\mathbb{L}_{x_1}(\Gamma \Rightarrow \Delta)$ is provable in the corresponding end-active LLS calculus.

Note that, in an end-active LLS, state variables might occur in the sequent and not in the relation set. Such formulae will remain inactive towards the leaves of the derivation and absorbed by the initial sequents in systems where weakening is admissible.

It is straightforward to extend the concepts of LLS and $\mathbb{T}\mathbb{L}_x$ in order to handle the extensions LNS_m and LNS_e: just add the relations $R_m \subseteq \text{SV} \times \text{SV}$ and $R_e \subseteq \text{SV} \times (\text{SV} \times \text{SV})$, respectively, and define

$$\begin{aligned}
\mathbb{T}\mathbb{L}_{x_n}^m(\Gamma_1 \Rightarrow \Delta_1 // \dots // \Gamma_n \Rightarrow \Delta_n) &= x_{n-1} R_m x_n, x_1 : \Gamma_1, \dots, x_n : \Gamma_n \Rightarrow x_1 : \Delta_1, \dots, x_n : \Delta_n \\
\mathbb{T}\mathbb{L}_{x_n}^e(\Gamma_1 \Rightarrow \Delta_1 // \dots // \Gamma_n \Rightarrow \Pi; \Omega \Rightarrow \Theta) &= x_{n-1} R_e(x_n, y_n), x_1 : \Gamma_1, \dots, x_n : \Sigma, y_n : \Omega \Rightarrow x_1 : \Delta_1, \dots, x_n : \Pi, y_n : \Theta
\end{aligned}$$

$$\begin{array}{c}
\frac{x_{n-1}Rx_n, X, x_n : A \Rightarrow Y \quad x_{n-1}Ry_n, X \Rightarrow Y, y_n : A}{x_{n-1}R_e(x_n, y_n), X, x_{n-1} : \Box A \Rightarrow Y} \mathbb{TL}_x^e(\Box_L^e) \\
\\
\frac{x_nR_e(x_{n+1}, y_{n+1}), X, y_{n+1} : B \Rightarrow Y, x_{n+1} : B}{x_{n-1}Rx_n, X \Rightarrow Y, x_n : \Box B} \mathbb{TL}_x^e(\Box_R^e) \quad \frac{x_nRx_{n+1}, X \Rightarrow Y, x_{n+1} : B}{x_{n-1}Rx_n, X \Rightarrow Y, x_n : \Box B} \mathbb{TL}_x^e(N) \\
\\
\frac{x_{n-1}R_e(x_n, y_n), X, y_n : \perp \Rightarrow Y}{x_{n-1}R_e(x_n, y_n), X \Rightarrow Y} \mathbb{TL}_x^e(M) \quad \frac{x_{n-1}R_e(x_n, y_n), X, x_n : A \Rightarrow Y \quad x_{n-1}Ry_n, X \Rightarrow Y, y_n : A}{x_{n-1}R_e(x_n, y_n), X, x_{n-1} : \Box A \Rightarrow Y} \mathbb{TL}_x^e(C)
\end{array}$$

Fig. 17. System LLS_e for non-normal labelled systems

$$\begin{array}{c}
\frac{x_nR_mx_{n+1}, X \Rightarrow Y, x_{n+1} : B}{x_{n-1}Rx_n, X \Rightarrow Y, x_n : \Box B} \mathbb{TL}_x^m(\Box_R^m) \quad \frac{x_{n-1}Rx_n, X, x_n : B \Rightarrow Y}{x_{n-1}R_mx_n, X, x_{n-1} : \Box B \Rightarrow Y} \mathbb{TL}_x^m(\Box_L^m) \\
\\
\frac{x_{n-1}R_mx_n, X \Rightarrow Y}{x_{n-1}Rx_n, X \Rightarrow Y} \mathbb{TL}_x^m(C) \quad \frac{x_{n-1}Rx_n, X \Rightarrow Y}{x_{n-1}R_mx_n, X \Rightarrow Y} \mathbb{TL}_x^m(N)
\end{array}$$

Fig. 18. LLS_m for monotone labelled systems

$$\frac{x_nR_mx_{n+1}, X \Rightarrow Y}{x_{n-1}Rx_n, X \Rightarrow Y} \mathbb{TL}_x^m(P) \quad \frac{x_nRx_{n+1}, X, x_{n+1} : \Sigma \Rightarrow Y, x_{n+1} : \Pi}{x_{n-1}R_mx_n, X, x_n : \Sigma \Rightarrow Y, x_n : \Pi} \mathbb{TL}_x^m(T)$$

Fig. 19. Labelled systems for extensions of monotonic modal logics

The corresponding LLS rules for these systems are depicted in Figs. 17, 18 and 19. Observe that this is a trivial generalization of the algorithm in [Goré and Ramanayake 2012], with the careful remark that, in the case of non-normal systems, the algorithm generates premisses that are weakened w.r.t. the ones presented in Fig. 17. Thus, Theorem 6.5 is also valid for all the LLS_m and LLS_e systems presented in this work. Finally, it is worth noticing that the definition of the mapping \mathbb{TL}_x for auxiliary nesting operators is the same as the respective final nesting operators.

6.2. Bipoles

In this section we exploit the above mentioned fact that LNS systems often have separate left and right introduction rules for modalities in order to present a systematic way of representing labelled line nested rules as *bipole clauses*. For that, we will use (focused) linear logic (LLF), not only because it extends the works in, e.g., [Miller and Pimentel 2013; Nigam et al. 2016], but also since this enables us to use the rich linear logic meta-level theory in order to reason about the specified systems. It is worth noticing, though, that our approach is general enough for specifying inference rules in other frameworks, like LKF ([Miller and Volpe 2015; Marin et al. 2016]). The set of *formulae* of LLF is given by the following grammar:

$$F ::= p \mid p^\perp \mid 1 \mid 0 \mid \top \mid \perp \mid F_1 \otimes F_2 \mid F_1 \wp F_2 \mid F_1 \& F_2 \mid F_1 \oplus F_2 \mid \exists x.F \mid \forall x.F \mid ?F \mid !F$$

The connectives $\perp, \top, \&, \wp, \forall, ?$ are taken to be *negative*, the connectives $1, 0, \otimes, \oplus, \exists, !$ are considered to be *positive*. The notions of negative and positive polarities are extended to formulae in the natural way by considering the outermost connective. Formulae are taken to be in negation normal form using the standard dualities, e.g., $(A \otimes B)^\perp \equiv A^\perp \wp B^\perp$. Linear logic (one-sided) *sequents* then are simply multisets of linear logic formulae. *Focused Linear Logic* LLF then adds a focusing mechanism to this structure. We refer the reader to [Girard 1987] for the rules of unfocused linear logic and to [Andreoli 1992; Miller and Pimentel 2013] for the focused versions.

6.2.1. Specifying sequents. We briefly recapitulate the basic concepts of the specification of sequent-style calculi in LLF from [Miller and Pimentel 2013]. Let obj be the type of object-level formulae and let $[\cdot]$ and $[\cdot]^\perp$ be two meta-level predicates on these, i.e., both of type $obj \rightarrow o$, where o is a primitive type denoting formulas. Object-level sequents of the form $B_1, \dots, B_n \Rightarrow C_1, \dots, C_m$ (where $n, m \geq 0$) are specified as the multiset $[B_1], \dots, [B_n], [C_1], \dots, [C_m]$ within the LLF proof system. The $[\cdot]$ and $[\cdot]^\perp$ predicates identify which object-level formulas appear on which side of the sequent – brackets down for left (useful mnemonic: $[\cdot]$ for “left”) and brackets up for right. Finally, binary relations R are specified by a meta-level atomic formula of the form $R(\cdot, \cdot)$.

6.2.2. Specifying inference sequent rules. Inference rules are specified by a re-writing clause that replaces the active formulae in the conclusion by the active formulae in the premises. The linear logic connectives indicate how these object level formulae are connected: contexts are copied ($\&$) or split (\otimes), in different inference rules (\oplus) or in the same sequent (\wp). For example, the specification of (a representative sample of) the rules of LLS_K is

$$\begin{array}{ll} (\wedge_l) & [x:A \wedge B]^\perp \otimes R(z, x)^\perp \otimes [x:A] \wp [x:B] \\ (\wedge_r) & [x:A \wedge B]^\perp \otimes R(z, x)^\perp \otimes [x:A] \& [x:B] \\ (\Box_R) & [x:\Box A]^\perp \otimes R(z, x)^\perp \otimes \forall y.([y:A] \wp R(x, y)) \\ (\Box_L) & [x:\Box A]^\perp \otimes R(x, y)^\perp \otimes [y:A] \wp R(x, y) \end{array}$$

where all the variables are bounded by an outermost existential quantifier.

The correspondence between focusing on a formula and an induced big-step inference rule is particularly interesting when the focused formula is a *bipole*.

Definition 6.6. A *monopole* formula is a linear logic formula that is built up from atoms and occurrences of the negative connectives, with the restriction that $?$ has atomic scope. A *bipole* is a positive formula built from monopoles and negated atoms using only positive connectives, with the additional restriction that $!$ can only be applied to a monopole.

Roughly speaking, bipoles are positive formulae in which no positive connective can be in the scope of a negative one. Focusing on such a formula will produce a single positive and a single negative phase. This two-phase decomposition enables the adequate capturing of the application of an object-level inference rule by the meta-level logic. For example, focusing on the bipole clause (\Box_R) will produce the derivation

$$\frac{\pi_1 \quad \pi_2 \quad \frac{\Psi; \Delta', [y:A], R(x, y) \uparrow}{\Psi; \Delta' \Downarrow \forall y.([y:A] \wp R(x, y))} [R \uparrow, \forall, \wp]}{\Psi; \Delta \Downarrow \exists A, x, z. [x:\Box A]^\perp \otimes R(z, x)^\perp \otimes \forall y.([y:A] \wp R(x, y))} [\exists, \otimes]$$

where $\Delta = [x:\Box A] \cup R(z, x) \cup \Delta'$, and π_1 and π_2 are, respectively,

$$\frac{}{\Psi; [x:\Box A] \Downarrow [x:\Box A]^\perp} I_1 \quad \frac{}{\Psi; R(z, x) \Downarrow R(z, x)^\perp} [\exists, I_1]$$

This one-step focused derivation will: (a) consume $[x:\Box A]$ and $R(z, x)$; (b) create a fresh label y ; and (c) add $[y:A]$ and $R(x, y)$ to the context. Observe that this matches *exactly* the application of the object-level rule $\mathbb{T}L_x(\Box_R)$.

When specifying a system (logical, computational, etc) into a meta-level framework, it is desirable and often mandatory that the specification is *faithful*, that is, one step of computation on the object level should correspond to one step of logical reasoning in the meta level. This is what is called *adequacy* [Nigam and Miller 2010].

Definition 6.7. A specification of an object sequent system is *adequate* if provability is preserved for (open) derivations, such as inference rules themselves.

Clearly not every sequent rule can be (adequately) specified in LLF. As an example, the rule $\mathbb{T}L_x^m(T)$ (Fig. 19) cannot be properly specified in our setting, since it lacks a principal formula. But it is

$$\begin{aligned}
(\Box_R^e) \quad & \lceil x : \Box B \rceil^\perp \otimes R(w, x)^\perp \otimes \forall y \forall z. (\lceil y : B \rceil \wp \lceil z : B \rceil \wp R_e(x, (y, z))) \\
(\Box_L^e) \quad & \lceil x : \Box A \rceil^\perp \otimes R_e(x, (y, z))^\perp \otimes (\lceil y : A \rceil \wp R(x, y)) \otimes (\lceil z : A \rceil \wp R(x, z)) \\
(C) \quad & \lceil x : \Box A \rceil^\perp \otimes R_e(x, (y, z))^\perp \otimes (\lceil y : A \rceil \wp R_e(x, (y, z))) \otimes (\lceil z : A \rceil \wp R(x, z))
\end{aligned}$$

Fig. 20. The LLF specification of the modal rules of LLS_{EC} for the logic EC.

straightforward to show that all other LLS rules derived from LNS systems presented in this paper can be adequately specified.

As an example, Fig. 20 shows adequate specifications in LLF of the labelled systems for the logic EC. These specifications can be used for automatic proof search as illustrated by the following theorem which is shown readily using the methods in [Miller and Pimentel 2013].

THEOREM 6.8. *Let L be a LLS system. A sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in L if and only if there is a finite $L_0 \subseteq L$ with \mathcal{L}_0 the theory given by the clauses of an adequate specification of the inference rules of L_0 such that $\mathcal{L}_0; \mathcal{R} \uparrow \lceil \Gamma \rceil, \lceil \Delta \rceil$ is provable in LLF.*

A prototype of theorem prover for LLS specified systems was implemented in <http://subsell.logic.at/nestLL/>. The system (called Poule for *ProOver for seqUent and Labelled systEms*) has an LLF interpreter that takes specified LLS rules (LLF clauses – the theory) and sequents and outputs a proof of the sequent, if it is provable. The prover is parametric in the theory, hence it profits from the modularity of the specified systems. Although the process of constructing LLS from LNS systems can be easily automated, the resulting system is not efficient. A direct implementation of LNS systems in Prolog, parametric on the modal axioms, can be found in <https://logic.at/staff/lellmann/lnsprover/>. We have no intention of comparing such implementations, since they are different in nature: a direct prover built from axioms is more adequate for proving *sequents*, while the meta-level prover based in LL is suitable for proving *properties*.

7. CONCLUDING REMARKS AND FUTURE WORK

Following [Masini 1992], in [Lellmann 2015] linear nested sequents were considered as an alternative presentation for modal proof systems. Since locality often entails modularity, this enabled modular presentations for different modal systems just by adding the local rules related to the new modalities to already defined linear nested sequent systems. In [Lellmann and Pimentel 2015] we continued the programme of representing modal proof systems in LNS, including suitable extensions of K, a simply dependent bimodal logic and some standard non-normal modal logics.

In this paper, we have generalised the works *op. cit.*, presenting local systems for a family of simply dependent multimodal logics as well as a large class of non-normal modal logics. All the proposed systems were proven sound and complete w.r.t. the respective sequent systems and, as a side effect, we proved that each LNS system presented in this work could be restricted to its end-active version. This enabled a notion of normal forms for LNS derivations, narrowing the proof search space and hence allowing the proposal of more efficient local proof systems. The possibility of restricting systems to their end-active versions also entails an automatic procedure for obtaining labelled sequent versions of LNS systems. Finally, we showed that the inference rules of such labelled systems can be seen as bipoles in linear logic.

There are at least three future research directions that could be taken from this work.

First, following the works in [Miller and Pimentel 2013; Nigam et al. 2016], it should be possible to use some of the meta-theory of linear logic to draw conclusions about the object-level LNS systems. For example, the problem of providing general procedures for guaranteeing cut admissibility for nested systems is still little understood. It would be interesting to investigate if it is possible to internalize some cut-elimination algorithms in nested systems into linear logic, and use focusing for assuring the adequacy of translations from object to meta-level.

Second, it appears that the labelled systems given in this work are somehow connected to the semantic labels, as established in [Goré and Ramanayake 2012] for Poggiolesi's CSL [Poggiolesi 2009] and Negri's G3GL [Negri 2005]. Since in this work we restrict nested systems to the linear case and we deal with normal as well as non-normal systems, it would be interesting to see to what extent our labels reflect the semantic models behind such logics. For that, we intend to explore the relationship between this work and the recent work in mixed sequent systems for non-normal logics in [Negri 2017].

Last but not least, concerning the construction of the LNS systems themselves, a natural next step would be the investigation of general methods for obtaining such systems from cut-free sequent systems, or even directly from Hilbert-style axioms.

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A. PROOFS OF CUT ELIMINATION

Since the calculi include the contraction rules we follow the standard method of eliminating the *multicut* rule

$$\frac{\Gamma \Rightarrow \Delta, A^n \quad A^m, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ mcut}$$

(with $n, m \geq 1$) instead of the standard cut rule. As usual, since the latter is a specific instance of the multicut rule, this implies cut elimination. For the sake of exposition we deviate slightly from standard terminology in the following way.

Definition A.1. The *main formula* of an application of a propositional rule or the modal rule T from Fig. 9 is the formula occurring in the conclusion with a greater multiplicity than in any of the premisses. In particular, the *main formulae* of an application of init or \perp_L are all formulae occurring in the conclusion. The *main formulae* of an application of a modal rule from Fig. 9 apart from T are all the formulae occurring in the conclusion. In an application of a structural rule, i.e., Weakening or Contraction, there are no main formulae.

Hence, e.g., the formula $\Box A$ in would be a main formula in the applications of rules D and D4 below left and centre, but not in the application of rule T below right.

$$\frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow} \text{ D} \quad \frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow} \text{ D4} \quad \frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow} \text{ T}$$

In the statement of the cut elimination theorem we write $\mathcal{G}_{\mathcal{L}}\text{mcut}$ for the calculus $\mathcal{G}_{\mathcal{L}}$ with the multicut rule mcut.

THEOREM A.2. *Let \mathcal{L} be the logic \mathbf{MA} with $\mathcal{A} \subseteq \{\mathbf{N}, \mathbf{C}, \mathbf{P}, \mathbf{D}, \mathbf{4}\}$ or one of the logics $\{\mathbf{M5}, \mathbf{MP5}, \mathbf{M45}, \mathbf{MP45}, \mathbf{MD45}, \mathbf{K45}, \mathbf{KD45}\}$. Then every derivation in $\mathcal{G}_{\mathcal{L}}\text{mcut}$ can be converted into a derivation in $\mathcal{G}_{\mathcal{L}}$ with the same endsequent.*

PROOF. The proof of elimination of multicut is reasonably standard by induction on the tuples (c, d) in the lexicographic ordering $<_{\text{lex}}$, where c is the *complexity* of the application of multicut, i.e., the number of symbols in the cut formula, and d is its *depth*, i.e., the sum of the depths of the derivations of the two premisses of the application of multicut.

So take a topmost application

$$\frac{\frac{\mathcal{D}_1}{\vdots} R_1 \quad \frac{\mathcal{D}_2}{\vdots} R_2}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ mcut}$$

of multicut in a derivation in $\mathcal{G}_{\mathcal{L}}\text{mcut}$. Assume that this application of multicut is of complexity c and depth d , that \mathcal{D}_1 and \mathcal{D}_2 are the derivations of the two premisses of this application, and that R_1 and R_2 are the two last applied rules in \mathcal{D}_1 and \mathcal{D}_2 respectively. Furthermore, assume that we have shown the statement for applications of multicut with complexity c' and depth d' such that $(c', d') <_{\text{lex}} (c, d)$.

If $d = 0$, then both R_1 and R_2 are one of the rules init or \perp_L and the conclusion of the multicut is obtained directly by one of these rules.

So suppose $d > 0$. We distinguish cases according to whether an occurrence of the cut formula was a main formula in the last applied rule in \mathcal{D}_1 and \mathcal{D}_2 respectively.

- (1) No occurrence of the cut formula is a main formula in R_1 . In this case R_1 is a structural rule, the rule T, or a propositional rule apart from init , \perp_L . This case is handled as usual by pushing the multicut into the premiss(es) of R_1 and applying the induction hypothesis on the depth of the

application of multicut. E.g., if R_1 is \vee_L , the derivation \mathcal{D}_1 ends in

$$\frac{\Gamma', B \Rightarrow \Delta, A^n \quad \Gamma', C \Rightarrow \Delta, A^n}{\Gamma', B \vee C \Rightarrow \Delta, A^n} \vee_L$$

From this we obtain a new derivation ending in

$$\frac{\frac{\Gamma', B \Rightarrow \Delta, A^n \quad \frac{\vdots \mathcal{D}_2}{A^m, \Sigma \Rightarrow \Pi}}{\Gamma', B, \Sigma \Rightarrow \Delta, \Pi} \text{mcut} \quad \frac{\frac{\Gamma', C \Rightarrow \Delta, A^n \quad \frac{\vdots \mathcal{D}_2}{A^m, \Sigma \Rightarrow \Pi}}{\Gamma', C, \Sigma \Rightarrow \Delta, \Pi} \text{mcut}}{\Gamma', B \vee C, \Sigma, \Sigma \Rightarrow \Delta, \Pi} \vee_L$$

Now the two applications of multicut have complexity c and depth less than d and we are done using the induction hypothesis.

- (2) At least one occurrence of the cut formula is a main formula in R_1 , but none of its occurrences is a main formula in R_2 . This case is analogous to the previous case, but pushing the multicut into the premiss(es) of R_2 instead of R_1 .
- (3) Some occurrences of the cut formula are main formulae both in R_1 and R_2 . In this case we have $c > 1$, since for $c = 1$ the cut formula A is a propositional variable or \perp , and since some of its occurrences are main formulae both in R_1 and R_2 we would have $d = 0$. So the rules R_1, R_2 must be propositional rules apart from init, \perp_L or modal rules. As usual we distinguish cases according to the last applied rules R_1, R_2 , and first apply *cross-cuts*, i.e., multicuts on the premiss(es) of R_1 and the conclusion of R_2 and vice versa to eliminate occurrences of the cut formula from the premisses of the two rules. These multicuts then have smaller depth and are eliminated using the induction hypothesis. Then we reduce the complexity of the multicut. Since the propositional cases are standard we only treat an exemplary case.

(a) $R_1 = \vee_R$ and $R_2 = \vee_L$. Then the derivations \mathcal{D}_1 and \mathcal{D}_2 end in

$$\frac{\Gamma \Rightarrow \Delta, B \vee C^{n-1}, B, C}{\Gamma \Rightarrow \Delta, B \vee C^n} \vee_R \quad \frac{B, B \vee C^{m-1}, \Sigma \Rightarrow \Pi \quad C, B \vee C^{m-1}, \Sigma \Rightarrow \Pi}{B \vee C^m, \Sigma \Rightarrow \Pi} \vee_L$$

From this we obtain derivations ending in

$$\frac{\Gamma \Rightarrow \Delta, B \vee C^{n-1}, B, C \quad \frac{B, B \vee C^{m-1}, \Sigma \Rightarrow \Pi \quad C, B \vee C^{m-1}, \Sigma \Rightarrow \Pi}{B \vee C^m, \Sigma \Rightarrow \Pi} \vee_L}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, B, C} \text{mcut}$$

and

$$\frac{\frac{\Gamma \Rightarrow \Delta, D \vee C^{n-1}, B, C}{\Gamma \Rightarrow \Delta, B \vee C^n} \vee_R \quad D, B \vee C^{m-1}, \Sigma \Rightarrow \Pi}{D, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{mcut}$$

with D either of the formulae C, B . These two applications of multicut have complexity c and depth less than d and hence are eliminated using the induction hypothesis. From the resulting derivations finally we obtain a derivation ending in

$$\frac{\frac{\Gamma, \Sigma \Rightarrow \Delta, \Pi, B, C \quad B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}{\Gamma, \Sigma, \Gamma, \Sigma \Rightarrow \Delta, \Pi, \Delta, \Pi, C} \text{mcut} \quad C, \Gamma, \Sigma \Rightarrow \Delta, \Pi}{\frac{\Gamma, \Sigma, \Gamma, \Sigma, \Gamma, \Sigma \Rightarrow \Delta, \Pi, \Delta, \Pi, \Delta, \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{Con}} \text{mcut}$$

Here the newly introduced applications of mcut have depth possibly greater than d but complexity less than c , and hence also are eliminated using the induction hypothesis.

- (b) $R_1 = M$: In this case the logic is $M\mathcal{A}$ for $\mathcal{A} \subseteq \{N, P, D, 4\}$ or one of $\{M5, MP5, M45, MP45, MD45\}$. So R_2 is one of $M, P, D, T, 4, D4, D5$. For the sake of brevity, in the following we only show the reductions of the multicuts, denoted by \leadsto . It is straightforward to check that the newly introduced multicuts are of the same complexity but lower depth, or of lower complexity than the original one.

i. $R_2 = (M)$:

$$\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \frac{B \Rightarrow C}{\Box B \Rightarrow \Box C} M}{\Box A \Rightarrow \Box C} \text{mcut} \leadsto \frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C} \text{mcut} \quad \frac{A \Rightarrow C}{\Box A \Rightarrow \Box C} M$$

ii. $R_2 = P$: similar to the previous case.

iii. $R_2 = D$: The case where $m = 1$ is as above. If $m > 1$ we only need to add some structural rules:

$$\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \frac{B, B \Rightarrow}{\Box B, \Box B \Rightarrow} D}{\Box A \Rightarrow} \text{mcut} \leadsto \frac{\frac{A \Rightarrow B \quad B, B \Rightarrow}{A \Rightarrow} \text{mcut} \quad \frac{A \Rightarrow}{A, A \Rightarrow} W}{\frac{A, A \Rightarrow}{\Box A, \Box A \Rightarrow} D} \text{Con} \quad \frac{\Box A, \Box A \Rightarrow}{\Box A \Rightarrow}$$

iv. $R_2 = T$:

$$\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \frac{\Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, \Box B^m \Rightarrow \Delta} T}{\Gamma, \Box A \Rightarrow \Delta} \text{mcut} \leadsto \frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \frac{\Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \text{mcut} \quad \frac{A \Rightarrow B \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \text{mcut} \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} T$$

v. $R_2 = 4$:

$$\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \frac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C} 4}{\Box A \Rightarrow} \text{mcut} \leadsto \frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \Box B \Rightarrow C}{\Box A \Rightarrow C} \text{mcut} \quad \frac{\Box A \Rightarrow C}{\Box A \Rightarrow \Box C} 4$$

vi. $R_2 = D4$: We consider the case that $m = 2$. The other cases are as above.

$$\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \frac{\Box B, B \Rightarrow}{\Box B, \Box B \Rightarrow} D4}{\Box A \Rightarrow} \text{mcut} \leadsto \frac{A \Rightarrow B \quad \frac{\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} M \quad \Box B, B \Rightarrow}{\Box A, B \Rightarrow} \text{mcut}}{\frac{\Box A, A \Rightarrow}{\Box A, \Box A \Rightarrow} D4} \text{mcut} \quad \frac{\Box A, \Box A \Rightarrow}{\Box A \Rightarrow} \text{Con}$$

vii. $R_2 = D5$: as for M

(c) $R_1 = N$:

i. $R_2 = M$: Similar to case 3(b)ii

ii. $R_2 = P$: As in the previous case.

iii. $R_2 = D$: The case where $m = 2$ is similar to the previous case. For $m = 1$ we have:

$$\frac{\frac{\Rightarrow A}{\Rightarrow \Box A} N \quad \frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow} D}{\Box B \Rightarrow} \text{mcut} \leadsto \frac{\frac{\Rightarrow A \quad A, B \Rightarrow}{B \Rightarrow} \text{mcut} \quad \frac{B \Rightarrow}{B, B \Rightarrow} W}{\frac{B, B \Rightarrow}{\Box B, \Box B \Rightarrow} D} \text{Con} \quad \frac{\Box B, \Box B \Rightarrow}{\Box B \Rightarrow}$$

- iv. $R_2 = T$: Similar to case 3(b)iv.
- v. $R_2 = 4$: Like case 3(b)v.
- vi. $R_2 = D4$: The case with $m = 2$ is like case 3(b)vi. If $m = 1$ again we need some structural rules:

$$\frac{\frac{\Rightarrow A}{\Rightarrow \Box A} \text{ N} \quad \frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow} \text{ D4}}{\Box B \Rightarrow} \text{ mcut} \quad \sim \quad \frac{\frac{\frac{\Rightarrow A}{\Rightarrow \Box A} \text{ N} \quad \Box A, B \Rightarrow}{B \Rightarrow} \text{ mcut} \quad \frac{B \Rightarrow}{B, \Box B \Rightarrow} \text{ W}}{\Box B, \Box B \Rightarrow} \text{ D4}}{\Box B \Rightarrow} \text{ Con}$$

vii. $R_2 = D5$: As for the previous case.

viii. $R_2 = C$: We have the following (substituting N for C in the last step if Γ is empty):

$$\frac{\frac{\Rightarrow A}{\Rightarrow \Box A} \text{ N} \quad \frac{A^m, \Gamma \Rightarrow B}{\Box A^m, \Box \Gamma \Rightarrow \Box B} \text{ C}}{\Box \Gamma \Rightarrow \Box B} \text{ mcut} \quad \sim \quad \frac{\frac{\Rightarrow A}{\Rightarrow \Box A} \quad \frac{A^m, \Gamma \Rightarrow B}{\Gamma \Rightarrow B} \text{ C}}{\Box \Gamma \Rightarrow \Box B} \text{ mcut}$$

- ix. $R_2 = CD$: As for the previous case.
- x. $R_2 = C4$: Similar to case 3(c)xiii.
- xi. $R_2 = CD4$: Similar to case 3(c)xiii.
- xii. $R_2 = K4$: Similar to case 3(c)xiii.
- xiii. $R_2 = K45$:

$$\frac{\frac{\Rightarrow A}{\Rightarrow \Box A} \text{ N} \quad \frac{\Box A^{m-k}, \Box \Gamma, A^k, \Sigma \Rightarrow B, \Box \Delta}{\Box A^m, \Box \Gamma, \Box \Sigma \Rightarrow \Box B, \Box \Delta} \text{ K45}}{\Box \Gamma, \Box \Sigma \Rightarrow \Box B, \Box \Delta} \text{ mcut} \quad \sim \quad \frac{\frac{\frac{\Rightarrow A}{\Rightarrow \Box A} \text{ N} \quad \Box A^{m-k}, \Box \Gamma, A^k, \Sigma \Rightarrow B, \Box \Delta}{\Box \Gamma, A^k, \Sigma \Rightarrow B, \Box \Delta} \text{ mcut} \quad \frac{\Box \Gamma, \Sigma \Rightarrow B, \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box B, \Box \Delta} \text{ K45}}{\Box \Gamma, \Box \Sigma \Rightarrow \Box B, \Box \Delta} \text{ mcut}$$

xiv. $R_2 = KD45$: Similar to case 3(c)xiii.

- (d) $R_1 = 4$: Since the rule 4 is a special case of each of C4, K4, and K45, here we only treat the cases not involving C, i.e., where R_2 is one of M, P, D, T, 4, D4. The case of D5 does not occur with the considered logics.

- i. $R_2 = (M)$: similar to case 3(b)i
- ii. $R_2 = P$:

$$\frac{\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \frac{B \Rightarrow}{\Box B \Rightarrow}}{\Box A \Rightarrow} \text{ mcut} \quad \sim \quad \frac{\Box A \Rightarrow B \quad B \Rightarrow}{\Box A \Rightarrow} \text{ mcut}$$

- iii. $R_2 = D$: The case where $m = 2$ is similar to the previous case. If $m = 1$ we have the following reduction, using the fact that whenever 4, $D \in \mathcal{A}$, then $G_{M, \mathcal{A}}$ contains the rule D4:

$$\frac{\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \frac{B, C \Rightarrow}{\Box B, \Box C \Rightarrow} \text{ D}}{\Box A, \Box C \Rightarrow} \text{ mcut} \quad \sim \quad \frac{\frac{\Box A \Rightarrow B \quad C, B \Rightarrow}{\Box A, C \Rightarrow} \text{ mcut}}{\Box A, \Box C \Rightarrow} \text{ D4}$$

iv. $R_2 = T$:

$$\frac{\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \frac{\Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, \Box B^m \Rightarrow \Delta} T}{\Gamma, \Box A \Rightarrow \Delta} \text{mcut}$$

$$\sim \frac{\Box A \Rightarrow B}{\Gamma, \Box A, \Box A \Rightarrow \Delta} \frac{\frac{\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \text{mcut}}{\Gamma, \Box A \Rightarrow \Delta} \text{Con}$$

v. $R_2 = 4$:

$$\frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \frac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C} 4}{\Box A \Rightarrow} \text{mcut} \quad \sim \quad \frac{\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \Box B \Rightarrow C}{\Box A \Rightarrow C} \text{mcut}$$

$$\frac{\Box A \Rightarrow C}{\Box A \Rightarrow \Box C} 4$$

vi. $R_2 = D4$: The case where $m = 1$ is similar to the previous case respectively case 3(d)vi. If $m = 2$ we have (similarly to case 3(d)iv):

$$\frac{\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad \frac{B, \Box B \Rightarrow}{\Box B, \Box B \Rightarrow} D4}{\Box A \Rightarrow} \text{mcut} \quad \sim \quad \frac{\Box A \Rightarrow B}{\Box A \Rightarrow B} \frac{\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4 \quad B, \Box B \Rightarrow}{\Box A, B \Rightarrow} \text{mcut}$$

$$\frac{\Box A, \Box A \Rightarrow}{\Box A \Rightarrow} \text{Con}$$

(e) $R_1 = 5$: Again, since the rule 5 is a special case of rule K45, here we only consider the cases not including C, i.e., where the logic is not K45 or KD45. The remaining cases are treated in case 3j. In this case R_2 then is one of M, P, D, 4, D4, D5.

- i. $R_2 = M$: Similar to case 3(b)vi.
- ii. $R_2 = P$: Similar to case 3(c)vi.
- iii. $R_2 = D$: Where $m = 1$ this is similar to case 3(d)iii, using that in this case the calculus also includes the rules D5 and 4. Where $n = 1$ and $m = 2$, this is similar to case 3(c)vi.
- iv. $R_2 = 4$: For $n = 2$ we have:

$$\frac{\frac{\Rightarrow A, \Box A}{\Rightarrow \Box A, \Box A} 5 \quad \frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B} 4}{\Rightarrow \Box B} \text{mcut} \quad \sim \quad \frac{\frac{\Rightarrow A, \Box A}{\Rightarrow \Box A, \Box A} 5 \quad \Box A \Rightarrow B}{\Rightarrow B} \text{mcut}$$

$$\frac{\Rightarrow B}{\Rightarrow B, \Box B} W$$

$$\frac{\Rightarrow B, \Box B}{\Rightarrow \Box B, \Box B} 5$$

$$\frac{\Rightarrow \Box B, \Box B}{\Rightarrow \Box B} \text{Con}$$

For $n = 1$ we have:

$$\frac{\frac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B} 5 \quad \frac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C} 4}{\Rightarrow \Box A, \Box C} \text{mcut} \quad \sim \quad \frac{\frac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B} 5 \quad \frac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C} 4}{\Rightarrow A, \Box C} \text{mcut}$$

$$\frac{\Rightarrow A, \Box C}{\Rightarrow \Box A, \Box C} 5$$

v. $R_2 = D4$: The case of $n = 1$ and $m = 2$ is similar to case 3(e)ii, that for $n = 2$ and $m = 1$ to case 3(c)vi. For $n = m = 2$ we have:

$$\frac{\frac{\Rightarrow A, \Box A}{\Rightarrow \Box A, \Box A} 5 \quad \frac{A, \Box A \Rightarrow}{\Box A, \Box A \Rightarrow} D4}{\Rightarrow} \text{mcut}$$

$$\sim \frac{\frac{\Rightarrow A, \Box A}{\Box A, \Box A \Rightarrow} \text{D4} \quad \frac{\Rightarrow A, A}{\Rightarrow \Box A, \Box A} \text{5} \quad \frac{A, \Box A \Rightarrow}{A \Rightarrow} \text{mcut}}{\Rightarrow A} \text{mcut}$$

The case of $n = m = 1$ is similar to cases 3(d)iii and 3(d)v, using the fact that in this case the calculus also includes the rules 4 and D5.

vi. $R_2 = \text{D5}$: Similar to the previous case.

(f) $R_1 = \text{D5}$: Again, we only consider the cases not including C, for the remaining case see case 3k. The only relevant cases then are that R_2 is one of M, P, D, 4, D4, D5.

i. $R_2 = \text{M}$: Similar to case 3(b)v:

$$\frac{\frac{A \Rightarrow \Box B}{\Box A \Rightarrow \Box B} \text{D5} \quad \frac{B \Rightarrow C}{\Box B \Rightarrow \Box C} \text{M}}{\Box A \Rightarrow \Box C} \text{mcut} \quad \sim \quad \frac{A \Rightarrow \Box B \quad \frac{B \Rightarrow C}{\Box B \Rightarrow \Box C} \text{M}}{A \Rightarrow \Box C} \text{mcut} \text{D5}$$

ii. $R_2 = \text{P}$: Similar the previous case.

iii. $R_2 = \text{D}$: The case of $m = 2$ is similar to the previous case, with additional structural rules. The case of $m = 1$ is similar to case 3(b)v, using that in this case the calculus also includes the rule D4..

iv. $R_2 = 4$: Similar to case 3(f)i.

v. $R_2 = \text{D4}$: Similar to case 3(f)iii.

vi. $R_2 = \text{D5}$: As in case 3(f)i.

(g) $R_1 = \text{C}$:

i. $R_2 = \text{P}$: Similar to case 3(b)ii, using that in this case also CD is in the rule set:

$$\frac{\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{C} \quad \frac{A \Rightarrow}{\Box A \Rightarrow} \text{P}}{\Box \Gamma \Rightarrow} \text{mcut} \quad \sim \quad \frac{\Gamma \Rightarrow A \quad A \Rightarrow}{\Box \Gamma \Rightarrow} \text{mcut} \text{CD}$$

ii. $R_2 = \text{D}$: Similar to the previous case.

iii. $R_2 = \text{T}$:

$$\frac{\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{C} \quad \frac{\Sigma, \Box A^{m-1}, A \Rightarrow \Pi}{\Sigma, \Box A^m \Rightarrow \Pi} \text{T}}{\Sigma, \Box \Gamma \Rightarrow \Pi} \text{mcut} \quad \sim \quad \frac{\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{C} \quad \frac{\Sigma, \Box A^{m-1}, A \Rightarrow \Pi}{\Sigma, \Box \Gamma, A \Rightarrow \Pi} \text{mcut}}{\Sigma, \Gamma, \Box \Gamma \Rightarrow \Pi} \text{mcut} \quad \frac{\Sigma, \Gamma, \Box \Gamma \Rightarrow \Pi}{\Sigma, \Box \Gamma, \Box \Gamma \Rightarrow \Pi} \text{T} \text{Con}$$

iv. $R_2 = 4$: See case 3(g)xi.

v. $R_2 = \text{D4}$: Similar to case 3(g)xi.

vi. $R_2 = \text{D5}$: See case 3(g)xiii.

vii. $R_2 = \text{C}$:

$$\frac{\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{C} \quad \frac{A^m, \Sigma \Rightarrow B}{\Box A^m, \Box \Sigma \Rightarrow \Box B} \text{C}}{\Box \Gamma, \Box \Sigma \Rightarrow \Box B} \text{mcut} \quad \sim \quad \frac{\Gamma \Rightarrow A \quad A^m, \Sigma \Rightarrow B}{\Box \Gamma, \Sigma \Rightarrow B} \text{mcut} \text{C}$$

viii. $R_2 = \text{CD}$: Similar to the previous case.

ix. $R_2 = C4$:

$$\begin{array}{c}
 \frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} C \quad \frac{\Box A^{m-k}, \Box \Sigma, A^k, \Omega \Rightarrow B}{\Box A^m, \Box \Sigma, \Box \Omega \Rightarrow \Box B} K4 \\
 \hline
 \Box \Gamma, \Box \Sigma, \Box \Omega \Rightarrow \Box B \quad mcut
 \end{array}$$

$$\sim \frac{\Gamma \Rightarrow A}{\Box \Gamma, \Box \Sigma, \Box \Omega \Rightarrow \Box B} C \quad \frac{\Box A^{m-k}, \Box \Gamma, \Box \Sigma, A^k, \Omega \Rightarrow B}{\Box \Gamma, \Box \Sigma, A^k, \Omega \Rightarrow B} mcut \\
 \hline
 \frac{\Gamma, \Box \Gamma, \Box \Sigma, \Omega \Rightarrow \Box B}{\Box \Gamma, \Box \Gamma, \Box \Sigma, \Box \Omega \Rightarrow \Box B} K4 \\
 \hline
 \Box \Gamma, \Box \Sigma, \Box \Omega \Rightarrow \Box B \quad Con$$

x. $R_2 = CD4$: Similar to case 3(g)ix.

xi. $R_2 = K4$: See case 3(g)ix.

xii. $R_2 = K45$: Similar to case 3(g)xiii.

xiii. $R_2 = KD45$: Similar to case 3(g)xi:

(h) $R_1 = C4$: Similar to case 3g.

(i) $R_1 = K4$: See case 3h and case 3c.

(j) $R_1 = K45$: This case only occurs for the logics K45 and KD45, limiting the possible cases to the following:

i. $R_2 = 4$: See case 3(j)ix.

ii. $R_2 = D4$: See case 3(j)x.

iii. $R_2 = D5$: See case 3(j)x.

iv. $R_2 = C$: See case 3(j)ix.

v. $R_2 = CD$: See case 3(j)x.

vi. $R_2 = C4$: See case 3(j)ix.

vii. $R_2 = CD4$: See case 3(j)x.

viii. $R_2 = K4$: See case 3(j)ix.

ix. $R_2 = K45$: We show the most interesting case, the remaining cases are similar.

$$\frac{\Box \Gamma, \Sigma \Rightarrow A, \Box A^{n-1}, \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box A^n, \Box \Delta} K45 \quad \frac{\Box A^{m-k}, \Box \Omega, A^k, \Theta \Rightarrow B, \Box \Pi}{\Box A^m, \Box \Omega, \Box \Theta \Rightarrow \Box B, \Box \Pi} K45 \\
 \hline
 \Box \Gamma, \Box \Sigma, \Box \Omega, \Box \Theta \Rightarrow \Box \Delta, \Box B, \Box \Pi \quad mcut$$

$$\sim \frac{\begin{array}{c} \vdots \mathcal{D}_1 \\ \Box \Gamma, \Sigma, \Box \Omega, \Box \Theta \Rightarrow A, \Box \Delta, \Box B, \Box \Pi \end{array} \quad \begin{array}{c} \vdots \mathcal{D}_2 \\ \Box \Gamma, \Box \Sigma, \Box \Omega, A^k, \Theta \Rightarrow \Box \Delta, B, \Box \Pi \end{array}}{\Box \Gamma, \Sigma, \Box \Omega, \Box \Theta, \Box \Gamma, \Box \Sigma, \Box \Omega, \Theta \Rightarrow \Box \Delta, \Box B, \Box \Pi, \Box \Delta, B, \Box \Pi} mcut \\
 \hline
 \frac{\Box \Gamma, \Box \Sigma, \Box \Omega, \Box \Theta, \Box \Gamma, \Box \Sigma, \Box \Omega, \Theta \Rightarrow \Box \Delta, \Box B, \Box \Pi, \Box \Delta, \Box B, \Box \Pi}{\Box \Gamma, \Box \Sigma, \Box \Omega, \Box \Theta \Rightarrow \Box \Delta, \Box B, \Box \Pi} K45 \\
 \hline
 \Box \Gamma, \Box \Sigma, \Box \Omega, \Box \Theta \Rightarrow \Box \Delta, \Box B, \Box \Pi \quad Con$$

where \mathcal{D}_1 is

$$\frac{\Box \Gamma, \Sigma \Rightarrow A, \Box \Delta}{\Box \Gamma, \Sigma, \Box \Omega, \Box \Theta \Rightarrow A, \Box \Delta, \Box B, \Box \Pi} K45 \quad \frac{\Box A^{m-k}, \Box \Omega, A^k, \Theta \Rightarrow B, \Box \Pi}{\Box A^m, \Box \Omega, \Box \Theta \Rightarrow \Box B, \Box \Pi} K45 \\
 \hline
 \Box \Gamma, \Sigma, \Box \Omega, \Box \Theta \Rightarrow A, \Box \Delta, \Box B, \Box \Pi \quad mcut$$

and \mathcal{D}_2 is

$$\frac{\Box \Gamma, \Sigma \Rightarrow A, \Box A^{n-1}, \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box A^n, \Box \Delta} K45 \quad \frac{\Box A^{m-k}, \Box \Omega, A^k, \Theta \Rightarrow B, \Box \Pi}{\Box \Gamma, \Box \Sigma, \Box \Omega, A^k, \Theta \Rightarrow \Box \Delta, B, \Box \Pi} mcut$$

- x. $R_2 = \text{KD45}$: Similar to the previous case.
- (k) $R_1 = \text{KD45}$: Similar to case 3j.

□